Antonine Physics A2

Topic 8 Further Mechanics

How to Use this Book

How to use these pages:

- This book intended to complement the work you do with a teacher, not to replace the teacher.
- Read the book along with your notes.
- If you get stuck, ask your teacher for help.
- The best way to succeed in Physics is to practise the questions.

There are many other resources available to help you to progress:

- Web-based resources, many of which are free.
- Your friends on your course.
- Your teacher.
- Books in the library.

This is an electronic book which you can download. You can carry it in a portable drive and access it from your school's computers (if allowed) as well as your own at home.

Further Mechanics builds on what you learned in mechanics in Physics 2. You may want to go back to revise this if you feel you have forgotten some of it. Further Mechanics goes on to explore different themes that can be explained by Newton's Laws.

We firstly study circular motion that will form the basis of the movement of stars, planets, and satellites. Circular motion concepts allow us to explain how cars go round corners, and how aeroplanes bank as they turn.

Then we will look at oscillations, when objects move with a regular to-and-fro movement. We will revise phase and explain resonance. From this we will go on to look at Simple Harmonic Motion, which explains how objects bounce on springs, and how pendulums swing.

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8.011 Motion in a Circle

The rules of **circular motion** help us to describe:

- movement of a car going round a corner,
- a tethered model aeroplane.
- the planets in their orbits.

We have so far applied Newton's Laws to **motion in a straight line**; the forces are pushing or pulling the object along in the direction of its travel. What happens if we put a force perpendicular to the direction of motion (*Figure 1*)?

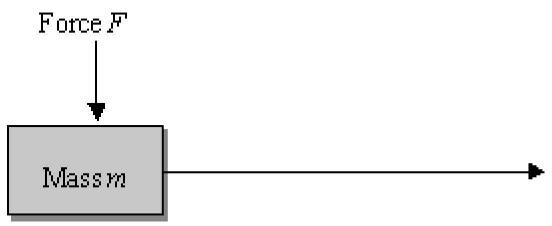


Figure 1 Applying a force to an object moving in a straight line

The path is **parabolic**.

Now let us suppose that the force applied at 90° to the object was always at 90°. This time the path would be **circular** (*Figure 2*).

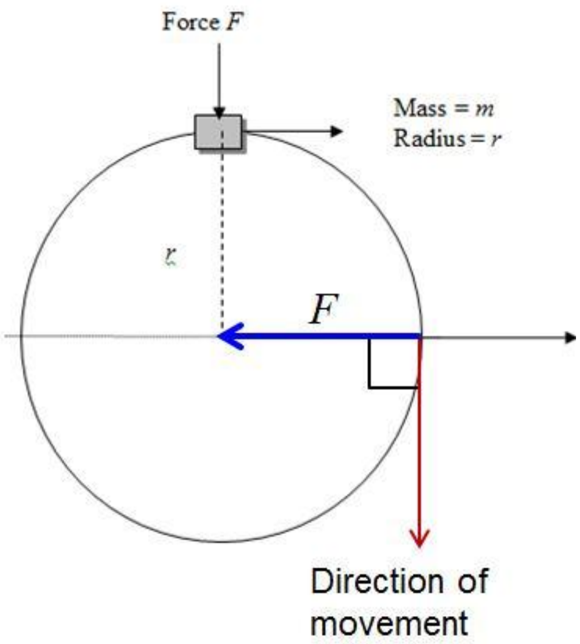


Figure 2 Applying a constant perpendicular force

In this case, the **linear speed** remains constant, but the direction is always changing. This means that there is a **change in velocity**, hence **acceleration**. Along with this come concepts that are important in circular motion, such as **angular velocity**. Note how the force is always towards the centre of the circle.

8.012 Angular Velocity

Consider an object going round in a circle of radius r (*Figure 3*).

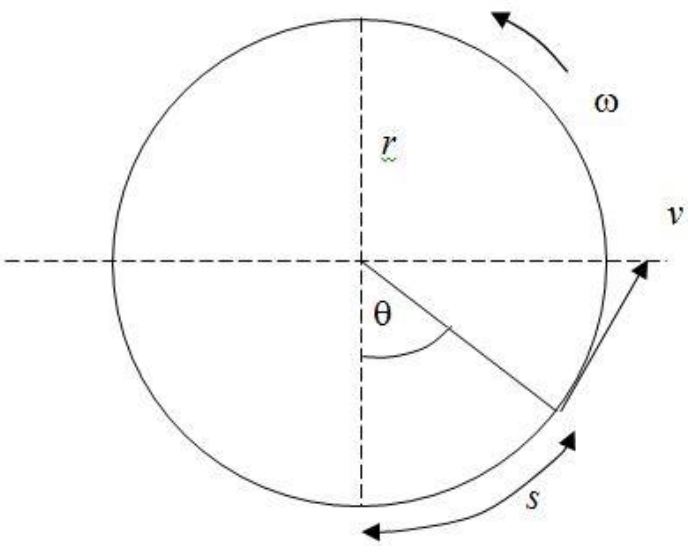


Figure 3 Object moving in a circular path of radius, r

We know that anything going round in a circle has a **constant speed** but a **changing velocity**. This is because the direction is constantly changing. If we alter the radius, the linear speed also changes, even though we have not changed the rate of turning. So, we have to think up another quantity that we can use to describe the rate of turning. We use **angular velocity**, physics code ω (omega, a Greek letter long 'ō'), how big an angle is turned in one second. We could use degrees per second, but instead we use another kind of angular measurement, the **radian**.

- One radian is the angle that subtends an arc whose length is the same as the radius.

$$\theta = \frac{s}{r} \ldots\ldots\ldots\ldots \textit{Equation 1}$$

- We can easily work out that 1 rad » 57 °.
- 1 revolution is **2π** radians.
- For small angles in radians, $\theta \approx \sin\theta \approx \tan\theta$. This is another reason why radians are so useful. It does not work for large angles in radians, nor does it work for degrees.
- In dimensional analysis the radian is a **dimensionless unit**. In some texts, you may see it missed out altogether, although here we will always include it.
- ω rad s^{-1} is the **angular velocity**.

$$\omega = 2\pi f \ldots\ldots\ldots\ldots \textit{Equation 2}$$

$\Rightarrow$ linear speed v

$$v = \omega r = 2\pi f r \ldots\ldots\ldots\ldots \textit{Equation 3}$$

- The direction of the velocity is **tangential**.

Make sure that you set your calculator to **radians**. It's up to you make sure that you know how to do this.

8.013 Frequency and Period

The **frequency** and **period** are linked to the angular velocity by these equations:

$$\omega = 2\pi f = \frac{2\pi}{T}$$

.... *Equation 4*

A common **bear-trap** is to fail to convert revolutions per minute to radians per second. Divide the rpm by 60, then multiply the answer by 2π.

8.014 Centripetal Acceleration

We need to distinguish between an object spinning on its axis, and an object moving in a circular path. We will consider the latter only. The former situation is part of **rotational dynamics**. You can read about the derivation of the relationship in any textbook, so we will not cover it here.

- Acceleration is always towards the centre of the circle and is given by:

$$a = \omega^2 r$$ .... *Equation 5*

- We can also express this in terms of frequency.

$$a = (2\pi f)^2 r = 4\pi f^2 r$$ .... *Equation 6*

- A very useful dodge here is that p^2 is approximately 10.
- We can write this as:

$$a = \frac{v^2}{r}$$

.... *Equation 7*

Where there is acceleration, there is a force. We call the force **centripetal force** (NOT centrifugal force!), which is described by the formula (*Equation 8*):

$$F = \frac{mv^2}{r}$$

.... *Equation 8*

The force acts towards the **centre** of the circle.

This is the circular motion version of Newton II, since $F = ma$ and $a = v^2/r$.

Suppose we were to whirl a stone around on a string, the forces would be governed by the relationship above. However, if the string were to snap, the stone would fly off in a straight line at a **tangent** to the circle (NOT straight out from the centre). The following are other examples that obey this relationship:

- Satellite orbiting the Earth (gravity provides the centripetal force)
- Vehicle going a bend (friction)
- Electron orbiting the nucleus (electrostatic attraction).

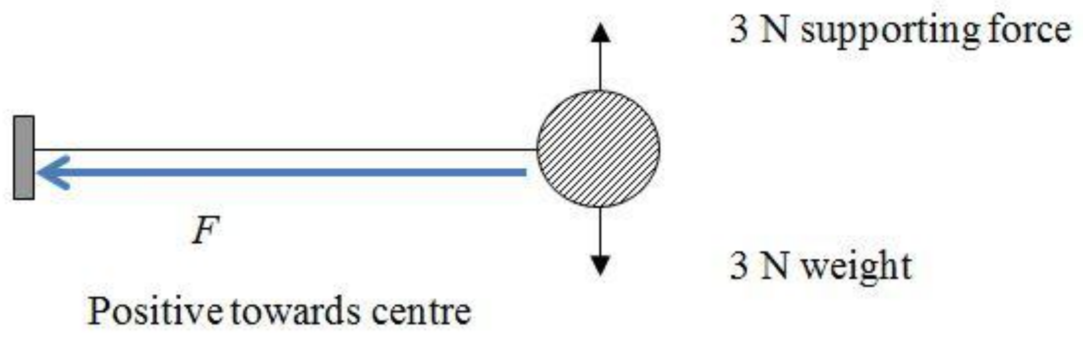
Problem solving strategy

1. State clearly what object is being considered.
2. Draw a free body sketch of the object.
3. Mark the weight of the object
4. Mark in any points where the object touches anything else and the forces involved.
5. Decide which direction you will call positive.
6. Apply Newton II ($F = ma$).

Worked example

A mass of 0.300 kg is moving in a circular path of radius 0.80 m on a friction free table. It is attached to a peg in the middle of the table in the centre of the circle. Draw the free body diagram of the mass and find the force exerted by the string on the mass when the mass is moving at a constant speed of 3.45 m s^{-1}.

Free body diagram:



$a = v^2/r$ = (3.45 m s^{-1})^2 ÷ 0.8 m = 14.9 m s^{-2}

$\Rightarrow F = ma$ = 0.300 kg × 14.9 m s^{-2} = **4.46 N**.

The string pulls the weight towards the middle of the circle.

8.015 Work done in Circular Motion

When an object is moving in circular motion, **no** work is done. This is because the centripetal force is **always at 90 degrees** to the direction of movement.

$$W = Fs\ \cos\theta \text{ ....} \quad \textit{Equation 9}$$

$$\cos 90 = 0$$

This is a favourite question in a multiple-choice paper.

8.016 Going Round the Bend

When a car goes around a corner, the centripetal force is caused by the **friction** of the tyres on the road. When you are in a car going round a right-hand bend, and you feel that you are being thrown out towards the left, in reality you are trying to go in a straight line, but the car is pushing on you, to make you go in a circle (*Figure 4*).

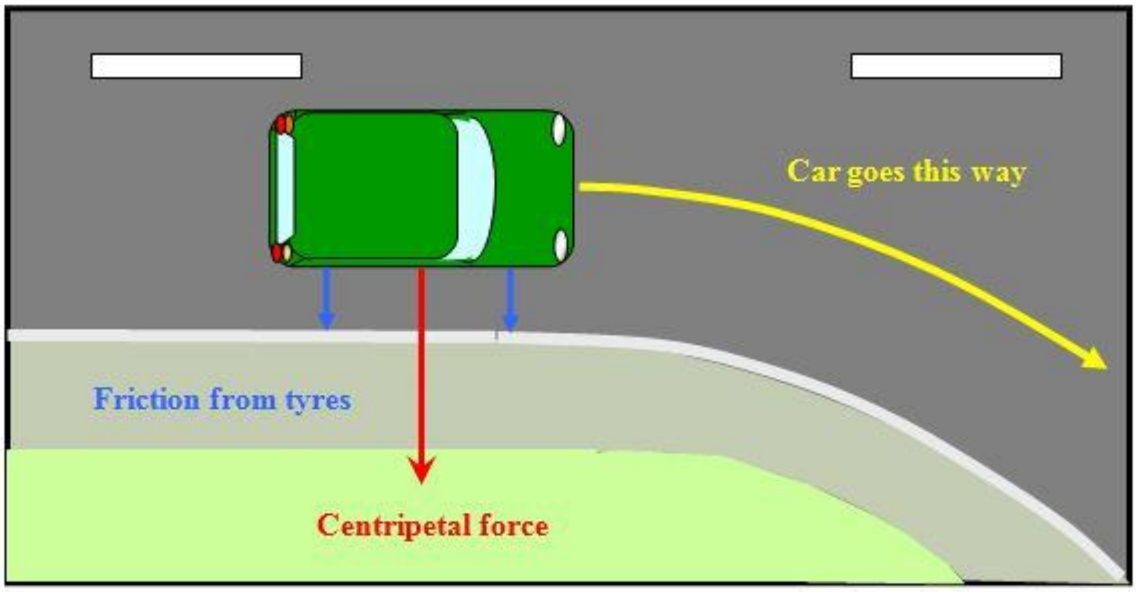


Figure 4 Car going round a bend

The centripetal force is given by:

$$F = \frac{mv^2}{r}$$ .... *Equation 10*

F is the friction of the tyres on the road. It can be worked out with a simple formula,

$$\mu = \frac{F}{N}$$ .... *Equation 11*

(μ is the **coefficient of friction**, F is the frictional force, and N is the **normal force**)

We can write

$$F = \mu N$$ .... *Equation 12*

Since N is the weight, we can rewrite this as

$$F = \mu mg$$ .... *Equation 13*

So, we can combine *Equations 10* and *13* to write *Equation 14:*

$$\mu mg = \frac{mv^2}{r}$$

.... *Equation 14*

The mass terms cancel out, so we can write:

$$\mu g = \frac{v^2}{r}$$

.... *Equation 15*

Therefore, for a particular radius of curve there is a maximum speed that depends on the coefficient of friction of the rubber on the road. Fortunately, the coefficient of friction for rubber is quite high, about 4.

8.017 Centrifugal force

Centrifugal force does NOT exist. It is a term that is used a lot by people to explain why, for example, you feel an outwards force when you are going round a corner in a car.

The answer, centrifugal force, is the most intuitive, but is not correct.

When a car goes round a corner, the friction of the tyres on the road causes the centripetal force. If the centripetal force were reduced to zero (e.g. the car it a patch of ice), the car would carry on in a straight line, tangential to the curved path it should have taken.

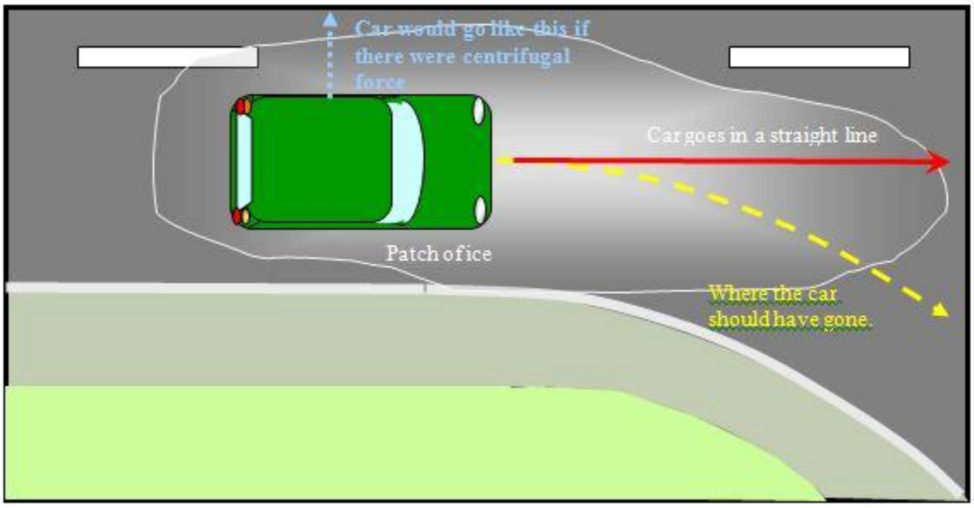


Figure 5 Car hitting a patch of ice carries on in a straight line

If there were centrifugal force (as a reaction force to the centripetal force), the car would slide at 90 ° to its path. This does not happen.

You will find the term *centrifugal force* used widely among engineers. Do not try to be the clever physicist who tells them that they are wrong.

Centrifugal force does not exist. Do NOT make any mention of it in the exam. It is a **physics error, and you will get no marks for that part of the question**.

Tutorial 8.01 Questions

8.01.1

What is the path of an object which is thrown horizontally and allowed to fall?

8.01.2

Why does an object going around in a circular path at a constant linear speed have acceleration?

8.01.3

A train is travelling at 50 m s^{-1} round a curve of radius 6000 m. What is its angular velocity?

8.01.4

A washing machine spins its drum at 1200 rpm. If the diameter of the drum is 35 cm, find:

(a) the angular velocity of the tub.

(b) the linear speed of the rim of the tub.

8.01.5

If $a = \omega^2 r$, show that $a = v^2/r$

8.01.6

In the discussion on Page 11, does the mass of the car matter?

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8.023 Vertical circular motion	8.024 Verification of Circular Motion

Banked Tracks

The maximum speed of a vehicle around a corner can be increased by **banking** the track. This is used widely on high-speed railway lines, as well as motor-racing tracks.

The diagram (*Figure* 6) shows a wagon going round a corner which is banked. (Sometimes this is called **super-elevation** or **cant**.)

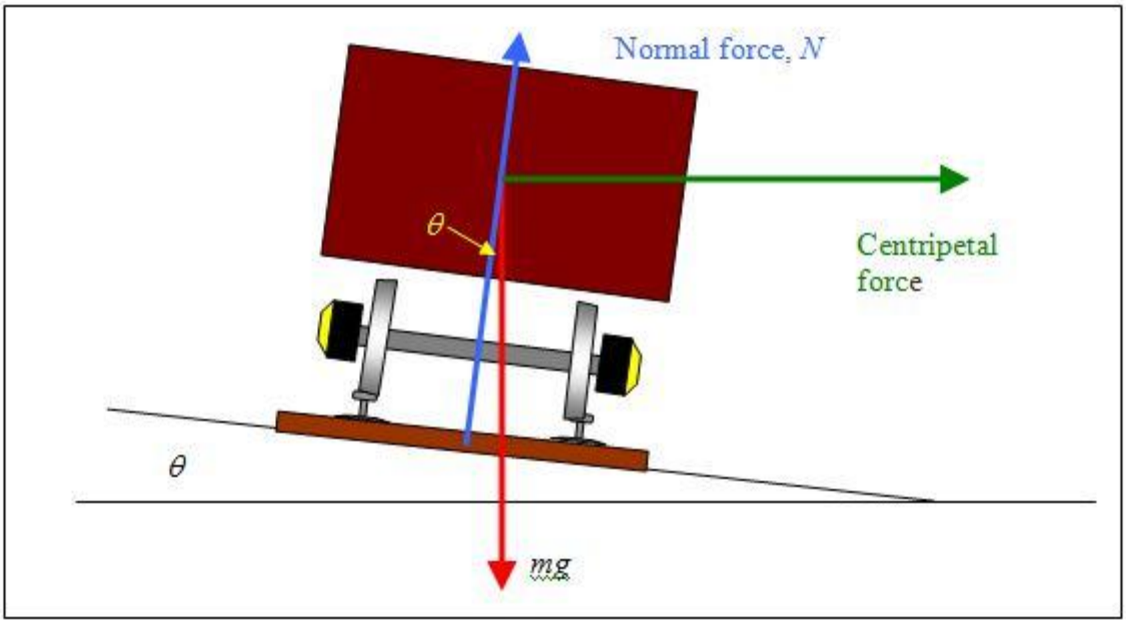

Figure 6 Railway wagon on a banked track

In this situation, the weight is the **perpendicular**, because it always acts vertically downwards and the centripetal force is at 90 degrees to the line of action of the weight. The centripetal force arrow has been made bigger for clarity.

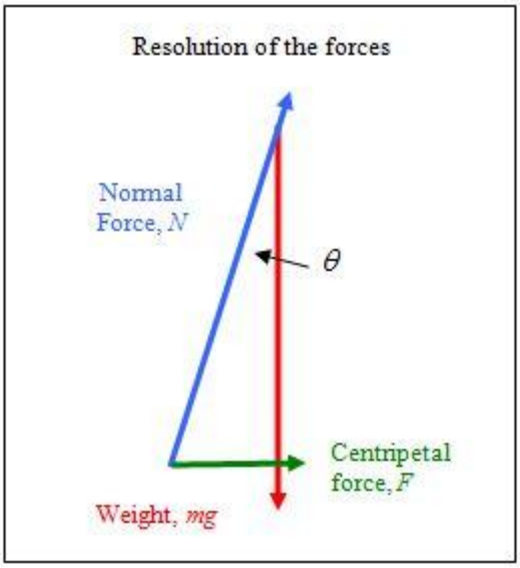

Figure 7 Resolution of the forces

The normal force is the **hypotenuse**. We can write an equation for the centripetal force:

$$F = N\ \sin\theta$$ .... *Equation 16*

And:

$$W = mg$$ .... *Equation 17*

The **normal force** is related to the weight by:

$$mg = N\ \cos\theta$$ .... *Equation 18*

This rearranges to:

$$N = \frac{mg}{\cos\theta}$$

.... *Equation 19*

We also know that:

$$N\sin\theta = \frac{mv^2}{r}$$

.... *Equation 20*

So, by substituting *Equation 19* into 20, we can say:

$$\frac{mg\sin\theta}{\cos\theta} = \frac{mv^2}{r}$$

.... *Equation 21*

The mass terms cancel out. Now, $\sin \div \cos = \tan$, so we can say:

$$g\tan\theta = \frac{v^2}{r}$$

.... *Equation 22*

This rearranges to give:

$$v^2 = gr\ \tan\theta$$ .... *Equation 23*

8.02.2 Banking in Aircraft

These ideas are also used to explain how aeroplanes bank when turning. Flaps on the wings (ailerons) lift one wing and push the other wing down. The aeroplane turns in a circle of radius that depends on the speed and the angle of bank.

Figure 8 The Red Arrows turning at an air display

The angle of bank is shown on the **attitude indicator**, a vital instrument for the pilot. It is shown in the picture below (*Figure 9*).

Figure 9 The attitude indicator on a light aeroplane

8.023 Vertical circular motion in a gravity field

Centripetal force is the main principle of many fairground and theme park rides. People pay good money to experience the changing forces. They apparently enjoy it rather like a baby in a baby-bouncer. And they pay good money for it, as well.

In the **wall of death machine**, people are spun around a drum which starts in the horizontal plane and then goes vertical – a big version of a washing machine. *Figure 10* shows the idea.

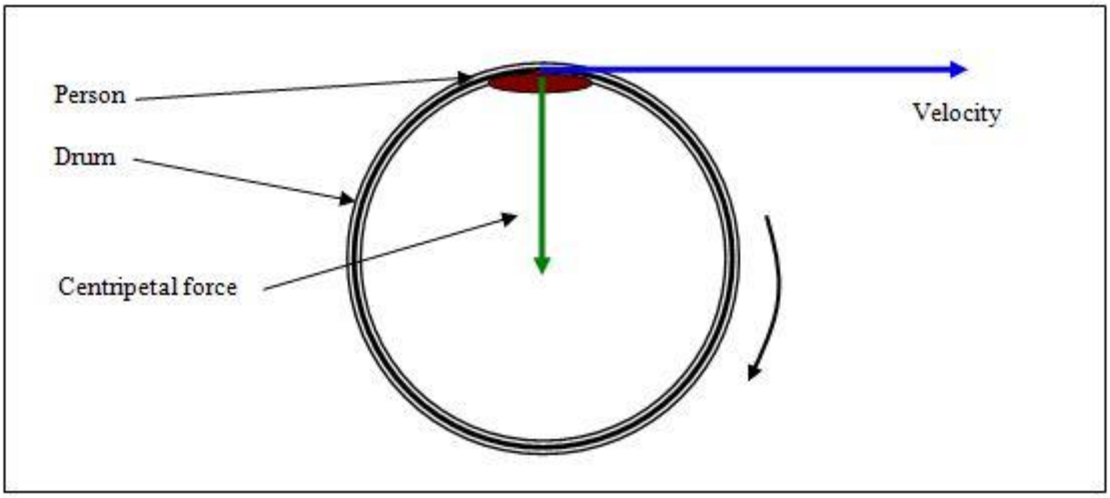

Figure 10 Wall of death machine

This is quite a difficult argument for students to accept as it is counter intuitive. Many try to explain it in terms of centrifugal force, which is wrong. Centrifugal force is NOT the reaction force to centripetal force; it does NOT exist.

The centripetal force, F, and the velocity vector, cause the wall to push on the participants with a **reaction force** R, who feel that they are being thrown outwards against the wall of the drum. (We could describe R as the "felt" force.) This is summed up in *Figure 11*.

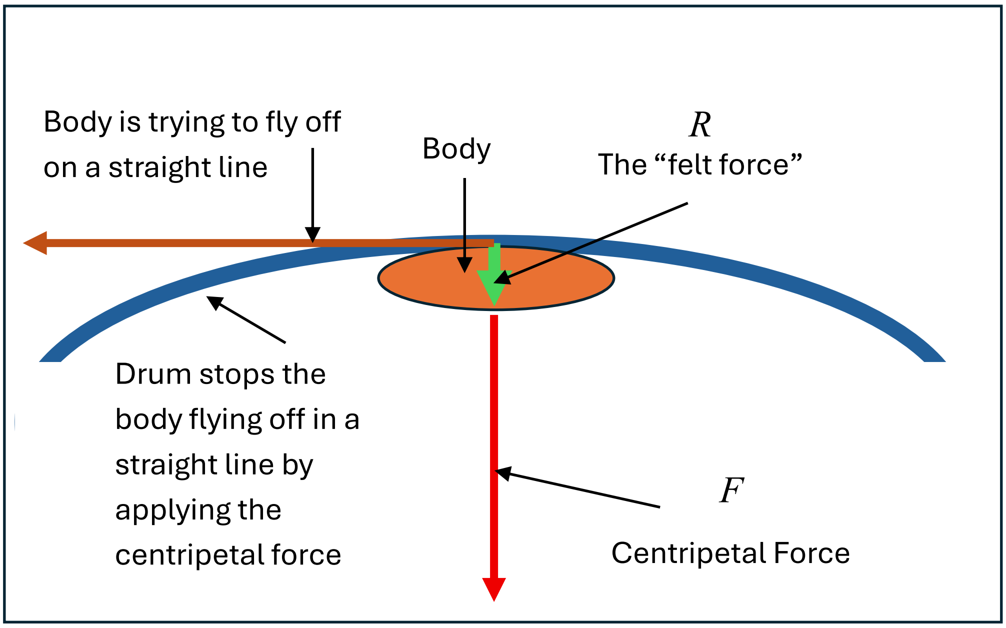

Figure 11 Forces acting on a body in a "Wall of Death machine"

When the machine is **horizontal**, the weight is acting at 90° to the force and is balanced by the floor, so can be disregarded:

$$F = \frac{mv^2}{r}$$ .... *Equation 24*

When the machine goes vertical, there are two important principles at work:

1. The weight mg is acting downwards all the time.
2. Both the **reaction force** and the **weight** vectors sum up to the **centripetal force**.

At the maximum height the weight and the reaction force are acting in the same direction, so the resulting force F is given by:

$$F = R + mg$$ .... *Equation 25*

We can rewrite this for any angle made by the radius and the centre line:

$$F = R + mg\ \sin\theta$$ .... *Equation 26*

Rearranging and substituting *Equation 24* for F:

$$R = \frac{mv^2}{r} - mg$$

.... *Equation 27*

The felt force = centripetal force – weight. This explains why you feel "lighter" when you go over the top of one of these machines.

At a certain speed, R becomes 0, so you feel weightless.

$$0 = \frac{mv^2}{r} - mg$$ .... *Equation 28*

Rearranging *Equation 28* gives:

$$\frac{mv^2}{r} = mg$$ .... *Equation 29*

As the m terms cancel out, and after further rearrangement of Equation29, we get:

$$v^2 = gr$$ .... *Equation 30*

At the **bottom** of the wheel the weight is acting in the opposite direction to the reaction force, so that:

$$F = R - mg$$ .... *Equation 31*

Therefore:

$$R = F + mg$$ .... *Equation 32n*

This is why you feel heavier going past the bottom of the contraption.

Centrifugal force should NOT be mentioned as part of this argument. Any mention of it will probably result in a physics error in the exam, and you may lose all the marks you would have got in that part of the question. However, any numerical answer would gain credit as an error carried forward, provided it is used correctly.

Aerobatic pilots know about felt force. As they go over the top of a circular manoeuvre, they feel weightless. Weightlessness training (*Figure 12*) for astronauts is carried out in converted airliners that follow a path like this. These aeroplanes are known as "vomit comets" for obvious reasons.

Figure 12 Weightlessness training for astronauts (Photo from NASA (Wikimedia Commons))

If they go even faster, they can experience "negative g". If this is excessive, it can do damage to their aeroplanes that is just as serious as positive g.

Flying a perfect vertical circle is not easy. The expression "going pear-shaped" has its origins in aerobatics. Many student stunt pilots' first attempts are decidedly pear-shaped.

8.024 Verification of Circular Motion

Your tutor will almost certainly get you to carry out an investigation into circular motion using apparatus like this (*Figure 13*).

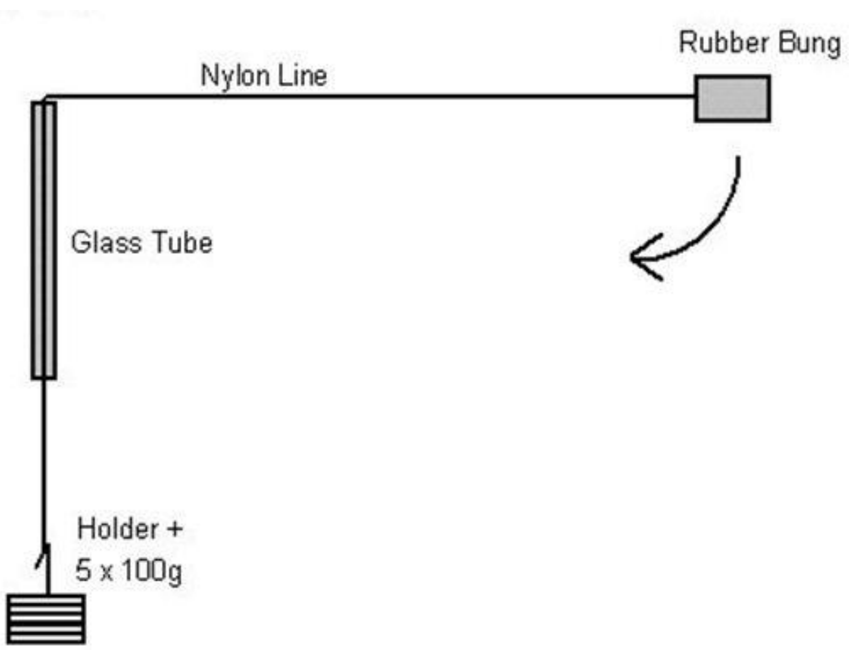

Figure 13 Experiment to verify principles of circular motion

- You start with a 100 g mass (0.98 N). You hold the tube and spin the rubber bung so that it holds the weight at a steady height. Too fast and the weight will go up into the tube; too slow and it will fall. You will have to practise as it's not easy to get it right first go!
- You need to time 50 revolutions of the bung and calculate the angular velocity ω.
- You measure the length of the string between the bung and the top of the tube, which will give the radius r.
- Now take readings for the other weights up to 5 N. Try to keep the radius r constant.

A plot of F against ω^2 should give a graph that shows **direct proportionality**. The gradient will be r. In reality, the results will be all over the place but repeat readings will reduce the uncertainty. You will be asked questions about the uncertainties.

This is a potentially hazardous experiment. A whirling bung hitting you can hurt; I know from my own experience. Always wear goggles and keep your group well away from other groups.

Tutorial 8.02 Questions

8.02.1

The radius of a railway track is 360 m.

It needs to take trains that are travelling at 18 m s^{-1}. What is the angle of bank that is needed?

8.02.2

An aeroplane is flying at constant height and constant speed of 50 m s^{-1}. The pilot banks to 30° to make a turn.

Use g = 9.8 m s^{-2}.

(a) What is the radius of the turn?

(b) What is the angular velocity of the turn?

(c) The pilot has a mass of 75 kg. What is the force acting on the pilot?

8.02.3

What would happen to the astronauts training in a plane as in *Figure 12* if the aeroplane went into negative g?

8.02.4

An aerobatic pilot makes a vertical circular loop of radius 1000 m. What speed does his machine need to travel at in order for him to feel zero g?

8.02.5

A wall of death machine has a radius of 12 m and rotates once every 6 seconds. Calculate:

a. The speed of rotation of the wheel

b. The centripetal acceleration of a person on the perimeter when the machine is horizontal.

c. The machine is now vertical. Calculate the support force, R, when the person of mass 72 kg is at the highest point.

d. ...and at the lowest point.

8.02.6

A stone of mass 0.50 kg is attached to an inextensible string and whirled in a vertical circle of radius 0.98 m at a constant speed of 7.0 m s^{-1}. Gravity g = 9.8 m s^{-2}

(a) Calculate the angular speed of the stone.

(b) Calculate the centripetal acceleration of the stone.

(c) The least tension in the string. (*Hint: think which way gravity acts*)

2. Oscillations and SHM

Tutorial 8.03 Oscillations and Resonance

All Syllabi

Contents

8.031 Free and Forced Vibrations

An **oscillation** is any to-and-fro movement. It can arise from:

- a swinging pendulum.
- a mass bouncing on a spring.
- a vibrating system.

We need to define some terms:

- **Cycle** – a complete to-and-fro movement.
- **Period** – time taken for a complete to-and-fro movement. It is given the Physics code T and measured in seconds (s).
- **Frequency** – how many cycles there are in a second. Physics code f and measured in Hertz (Hz).

$$f = \frac{1}{T}$$

.... *Equation 33*

8.032 Phase

Suppose we have two oscillators swinging at the same frequency. If they are swinging in step, we say that they **are in phase**. If they are swinging so that one reaches its maximum displacement to the left at the same time as the other reaches its maximum displacement to the right (*Figure 14*), we say that they are 180 ° (or π rad) **out of phase**.

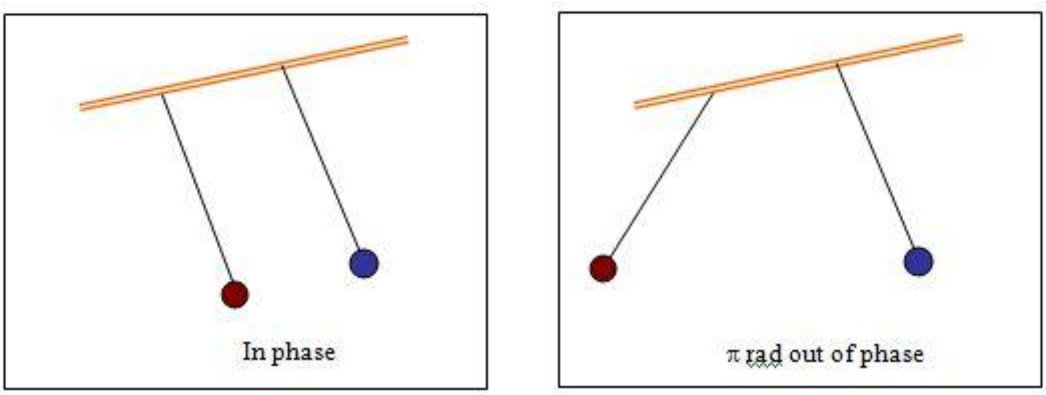

Figure 14 Phase differences between oscillators

Of course we can have any number of degrees out of phase. The oscillator that gets to a particular point first is **leading**. The other one is **lagging**.

We can show the displacement-time graph of two oscillators that are out of phase, but with the same frequency (*Figure 15*).

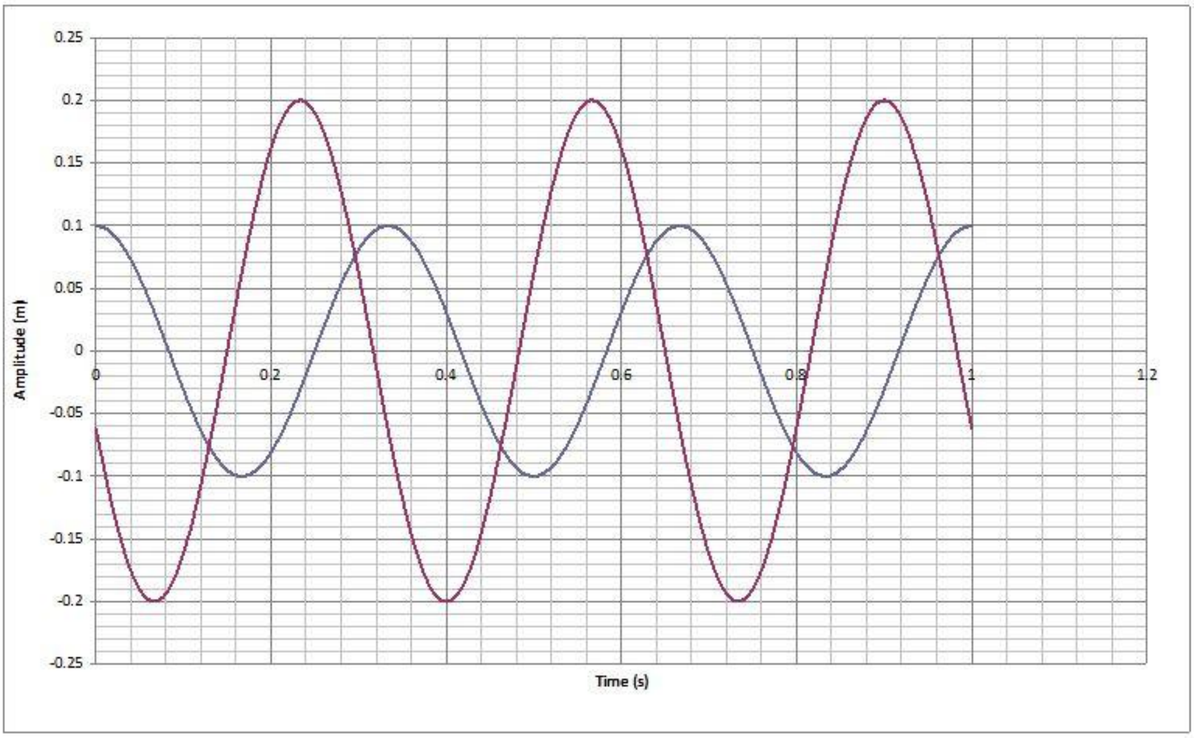

Figure 15 Oscillators of same frequency but out of phase.

If the frequencies are different, the phase relationship changes all the time (*Figure 16*).

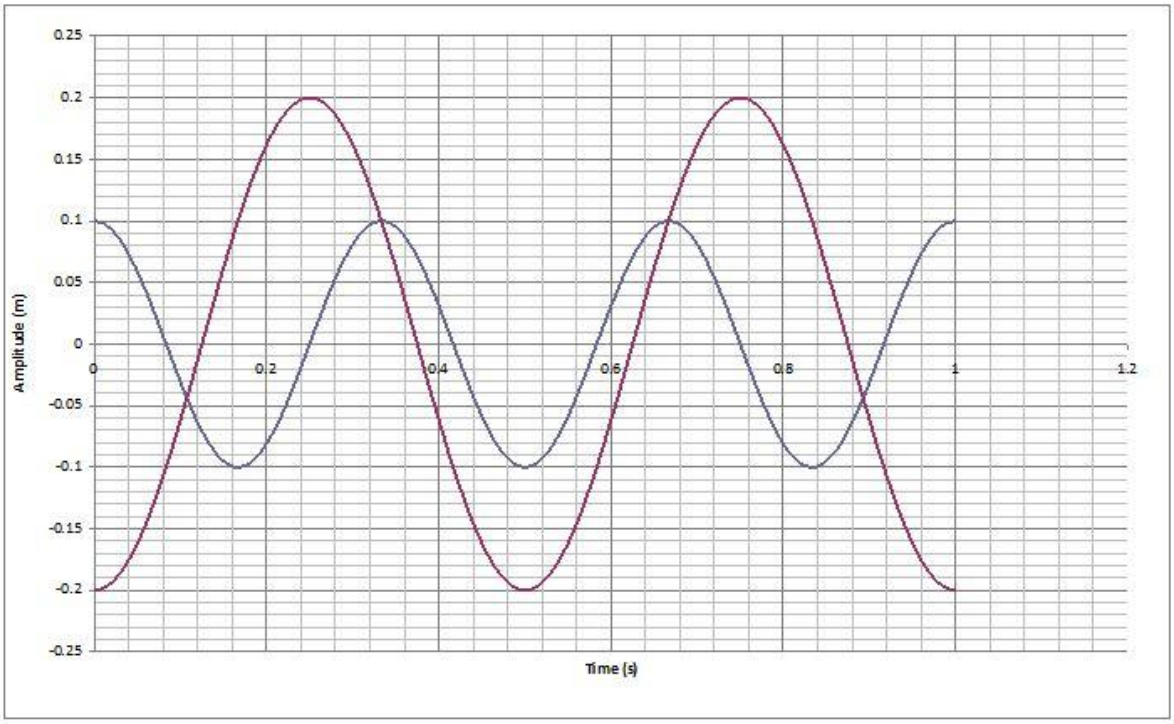

Figure 16 Phase relationship of two oscillators of different frequencies

8.033 Free and Forced Oscillations

If we swing a pendulum at let it swing freely, it will swing at its **natural frequency**. The same will apply to a mass bouncing up and down on a spring.

If we try to make the oscillator oscillate, we apply a **forcing frequency**. An example of this is the push we give to a child on a swing.

8.034 Resonance

If the forced vibrations have the same frequency as the natural frequency, the amplitude of the oscillations will get very large. We can show this with our child on the swing. If we apply the push at the same point of the swing every time, the child swings higher and higher. We call this situation **resonance**.

We can demonstrate resonance in the lab in several ways including:

- Barton's pendulums
- A mass on a spring being forced up and down with a vibration generator (*Figure 17*).

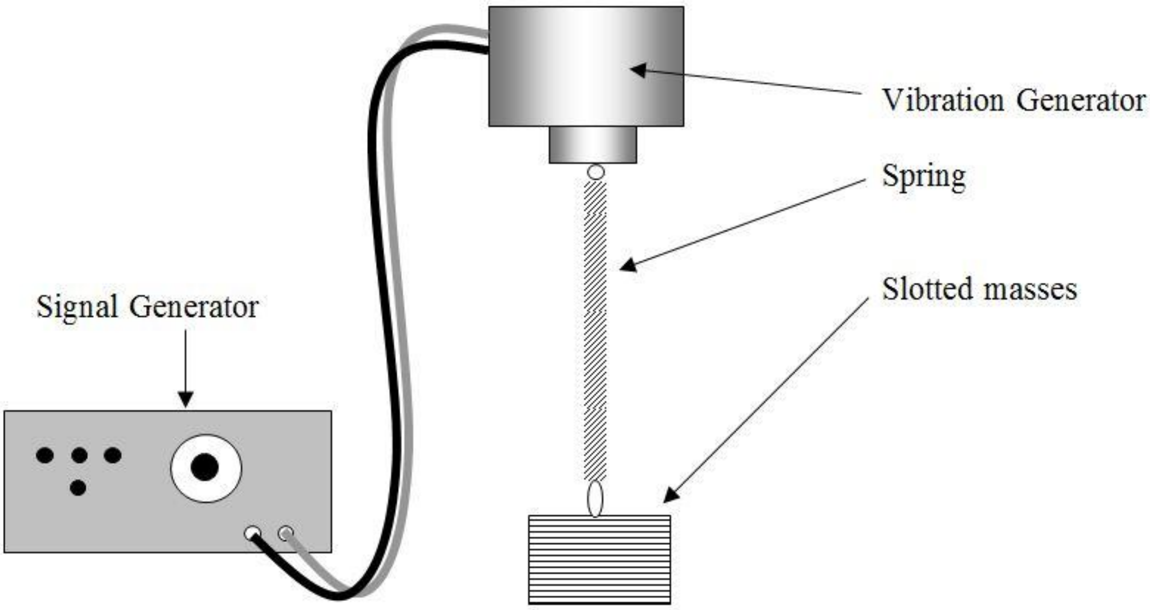

Figure 17 Demonstrating resonance with a vibration generator

If we alter the frequency, we see that the mass bounces with varying amplitude. However, at the resonant frequency, the amplitude gets very large. It is not unknown for the masses to fly off! Typically, the resonant frequency of this kind of system is about 1.5 Hz.

Another demonstration is to show **Barton's Pendulums** (*Figure 18*). It was named after Edwin Henry Barton (1858 – 1925), Senior Lecturer in Physics at University College, Nottingham. It consists of a number of pendulums of different lengths which are mounted from a string as shown:

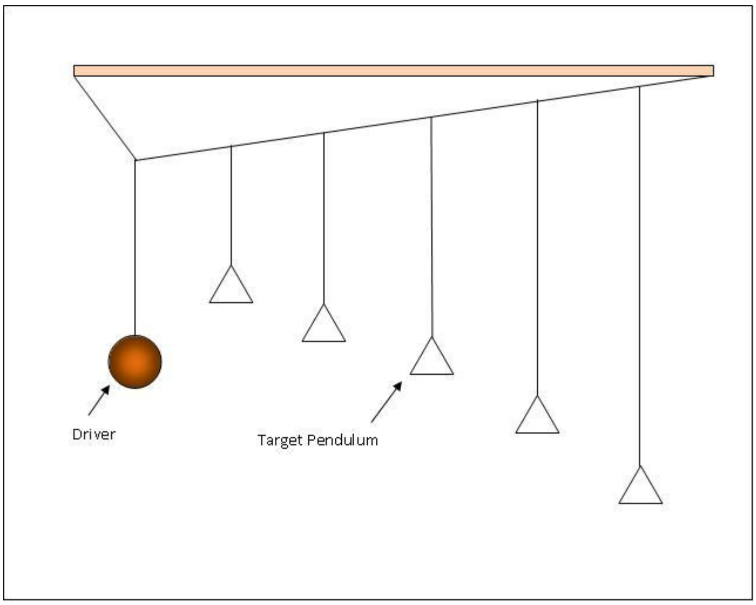

Figure 18 Barton's Pendulums

The apparatus demonstrates the **phase** as well as the amplitude of the oscillations. **Phase difference** describes how much oscillations are "out of step". The **driver** pendulum is set swinging, applying a torque to the string. This in turn sets the others swinging:

- The pendulums with the shorter strings than the target pendulum swing in phase with the driver.
- The **target pendulum** has a string the same length as the driver. It swings with the largest amplitude, as its natural frequency is the same as that of the driver. It is out of phase with the driver by $\pi/2$ rad.
- The pendulums with the longer strings oscillate π rad out of phase.

Therefore:

- If the forcing frequency is less than the natural frequency, the oscillators and the driver move in phase.
- If the forcing frequency is the same as the natural frequency, the driver leads the oscillator by 90 degrees (or $\pi/2$ rad)
- If the forcing frequency is bigger than the natural frequency, the oscillators oscillate in anti-phase.

In the exam, you may be asked to describe resonance. Remember to include discussion about phase.

If we plot a graph of amplitude against frequency (*Figure 19*), we see a very large peak. It occurs at the **resonant frequency**, which we give the code f_0. When considering the resonant frequency of strings and columns of air, we often call this the **fundamental frequency**.

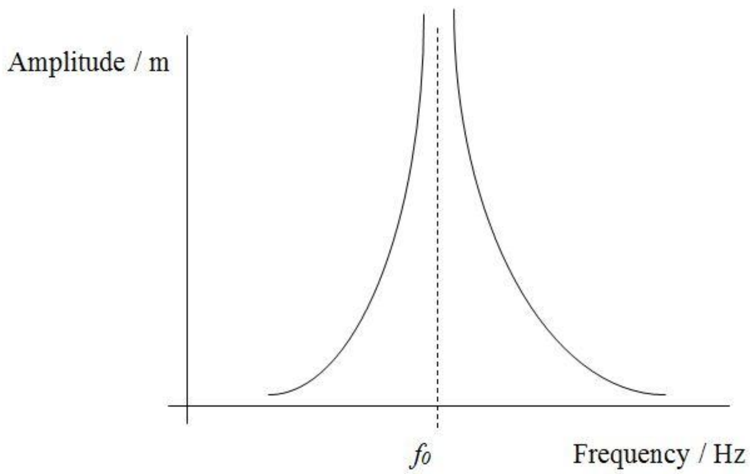

Figure 19 Graph showing resonant peak in amplitude

Resonance has many uses:

- In order to sound heavy church bells (which may have masses of several tonnes), bell ringers swing them at the resonant frequency of the bell in its carriage. They cannot swing them at any other rate.
- Resonance of strings at their fundamental frequency and multiples of them give us musical sounds. Wind instruments are sounded by making a column of air resonate by either blowing a whistle or a raspberry (an *embouchure*) at one end.
- Resonance of electrons makes radio waves and allows them to be received.

Resonance can also be a nuisance or even dangerous:

- Panels in a bus rattling.
- Resonance in a car suspension needs to be damped. If the dampers (shock absorbers) are not working properly, the car could go out of control.
- A suspension bridge started rocking in a wind at its resonant frequency. Its oscillations got so large that the deck collapsed into the sea.
- Marching troops are ordered to "break step" when crossing bridges.
- Resonance in an aircraft can make the machine difficult to control or even destroy it.

8.035 Damping

The amplitude of resonant oscillations can be reduced by **damping**.

- **Light** damping reduces oscillations slowly.
- **Heavy** damping reduces oscillations quickly.
- **Critical** damping stops the oscillation within one cycle.

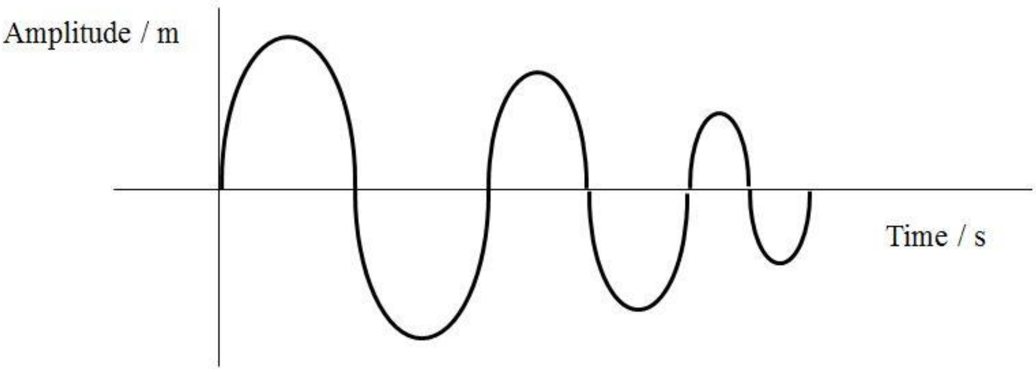

Figure 20 Amplitude-time graph showing light damping

The graph above shows light damping. Question 8.03.6 asks you to consider heavy and critical damping.

Over-damped systems do not oscillate**.** They take a long time to return to the equilibrium position. An example is the return spring on a door. The graph looks like this (*Figure 21*).

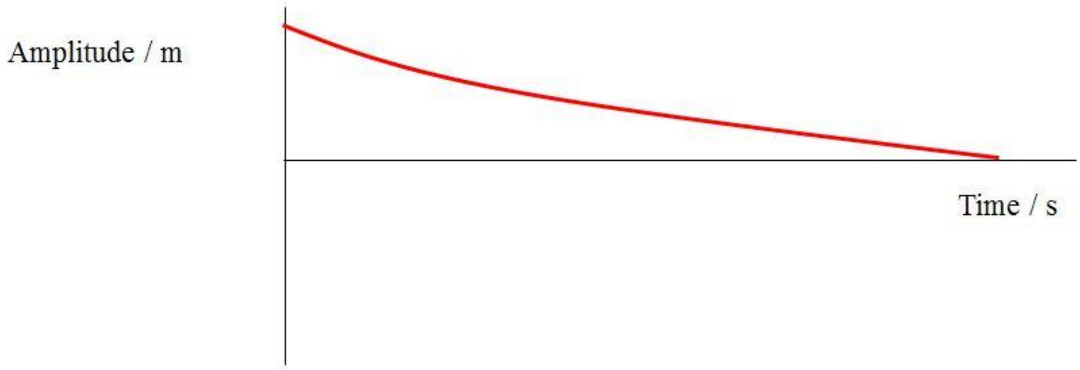

Figure 21 Over damping.

The graph below (*Figure 22*) shows the effect on amplitude of damping. The resonant frequency reduces slightly.

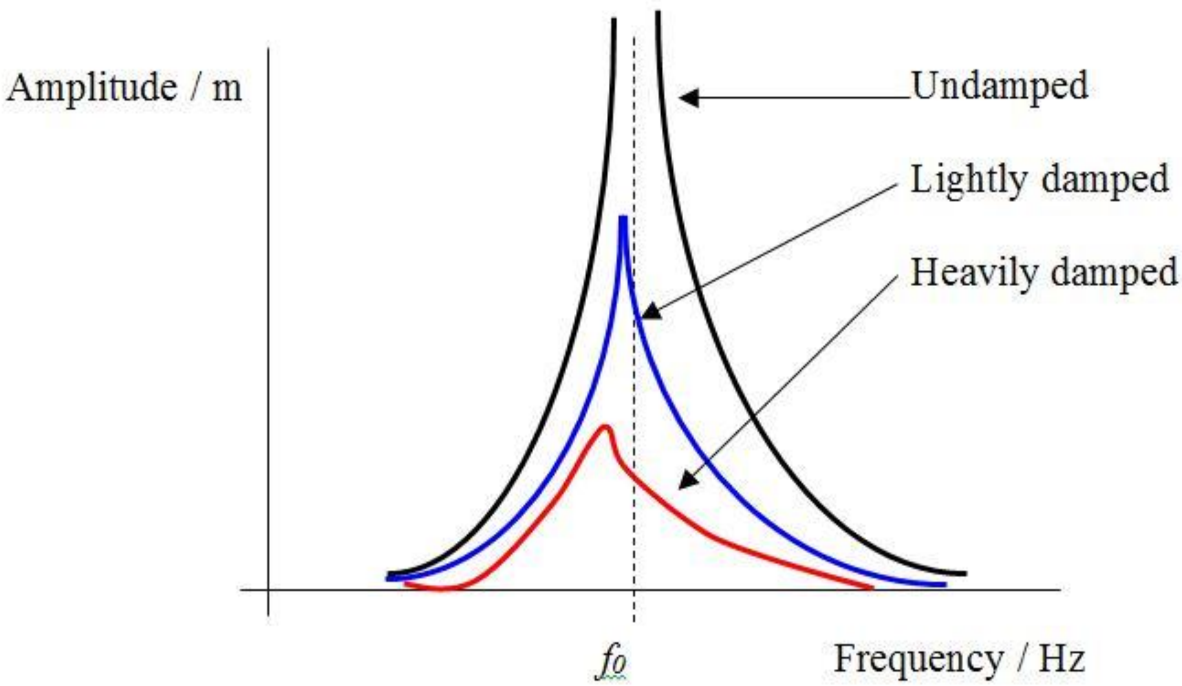

Figure 22 The effect of damping on resonant frequency

8.036 Damping and Period

When a damping occurs, the period of the damped oscillation **remains the same**, as shown in this diagram (*Figure 23*).

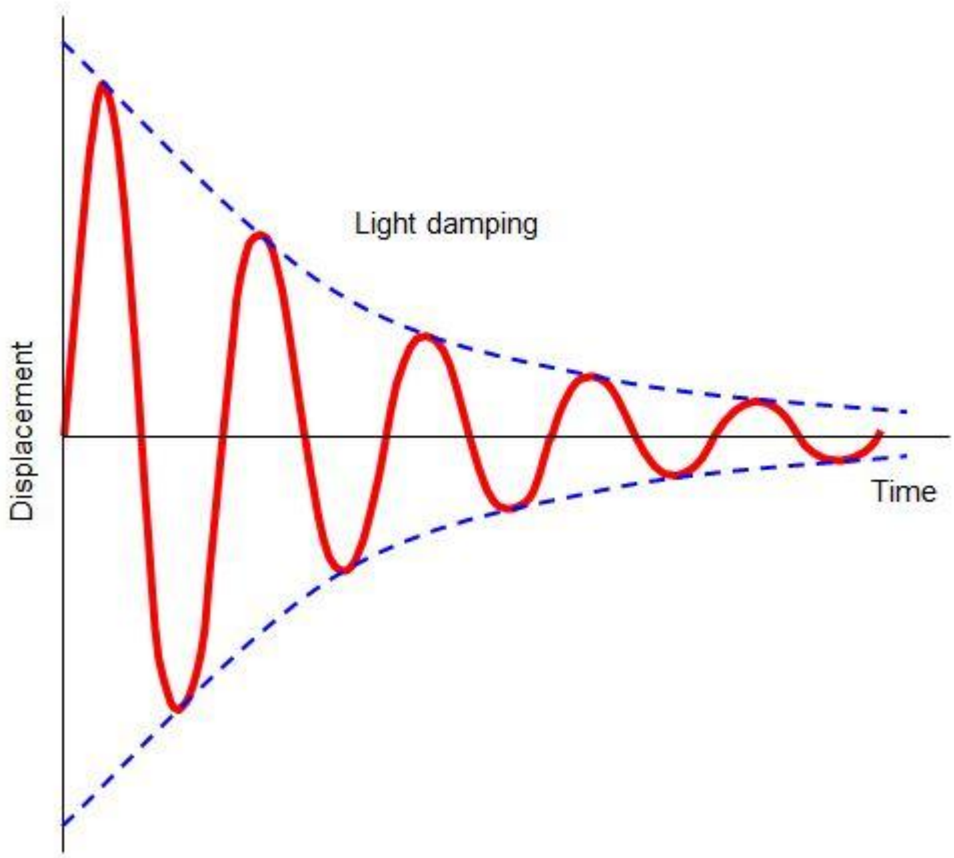

Figure 23 The period of a damped oscillation remains the same.

When damping occurs, the amplitude decays as a **constant fraction** of the amplitude of the previous oscillation. So, if the first oscillation has an amplitude of 100 % and the fraction is 0.9, the next oscillation has 90 %, the next 81 %, and so on. As with any system that decays by a constant ratio, natural logarithms are involved. Therefore, it is sometimes called the **logarithmic decrement**, and you will study the relationships at university (*Figure 24*).

$$\delta = \frac{1}{n} \ln \frac{x(t)}{x(t+nT)}$$

Figure 24 Logarithmic decrement equation (Image from Wikimedia Commons)

The symbol δ (delta, a Greek lower-case letter 'd') is the physics code for the decrement. $x(t)$ is the amplitude at time t, while $x(t + nT)$ is the amplitude a whole number of periods later. n is a whole number. This is not on the A level syllabus.

8.037 Plastic Deformation and Oscillation

So far, we have assumed that the oscillators have obey Hooke's Law and showed perfectly elastic deformation. But what happens to the way the oscillators behave if they exceed the elastic limit?

One example is what happens with a **climbing rope**. An ordinary rope would slow the climber down in a very short time, causing serious (if not life-threatening) injury. The speed-time graph would be like this (*Figure 25*)

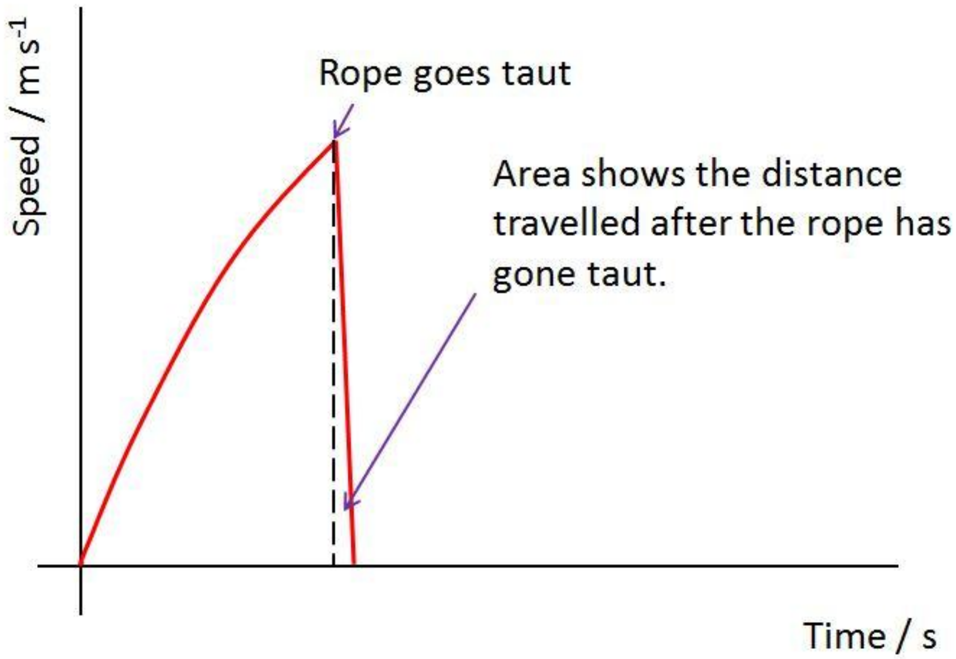

Figure 25 Why you should not use ordinary rope when rock climbing

The distance travelled is very small, so the body undergoes a very high value of deceleration. The energy will be dissipated as heat. The climber will not bounce up and down. Instead, he will be seriously injured.

Suppose you used a bungee rope instead. The motion of the climber would be as in *Figure 26*:

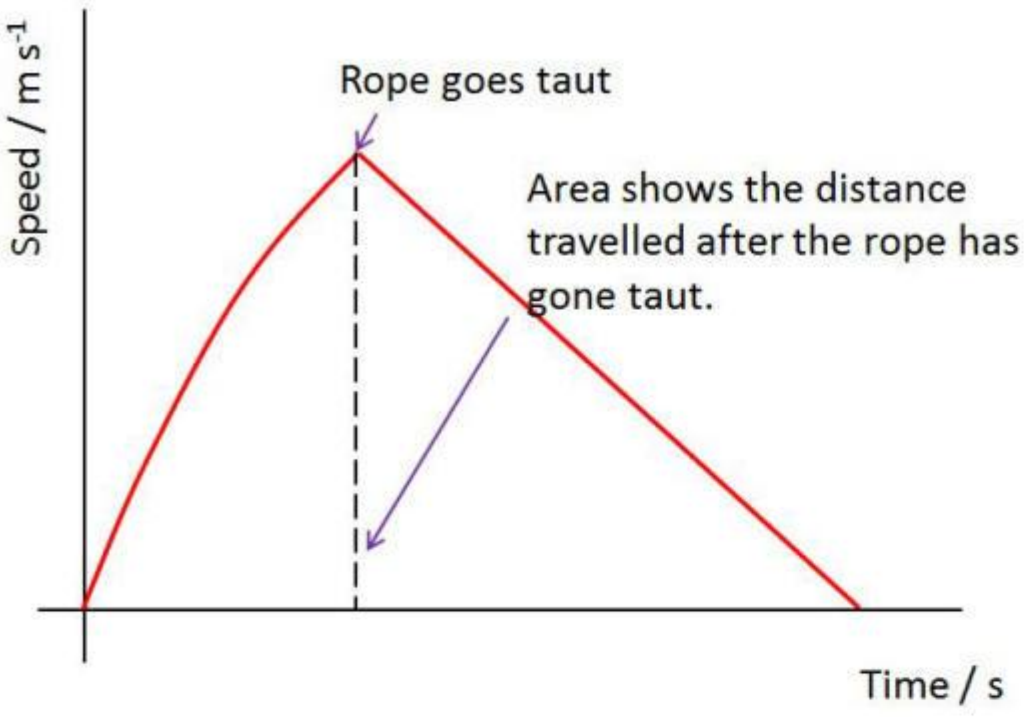

Figure 26 A bungee rope would result in lower rate of deceleration

The climber will not suffer any injury after the fall, but would bounce up and down, as shown in this graph (*Figure 27*).

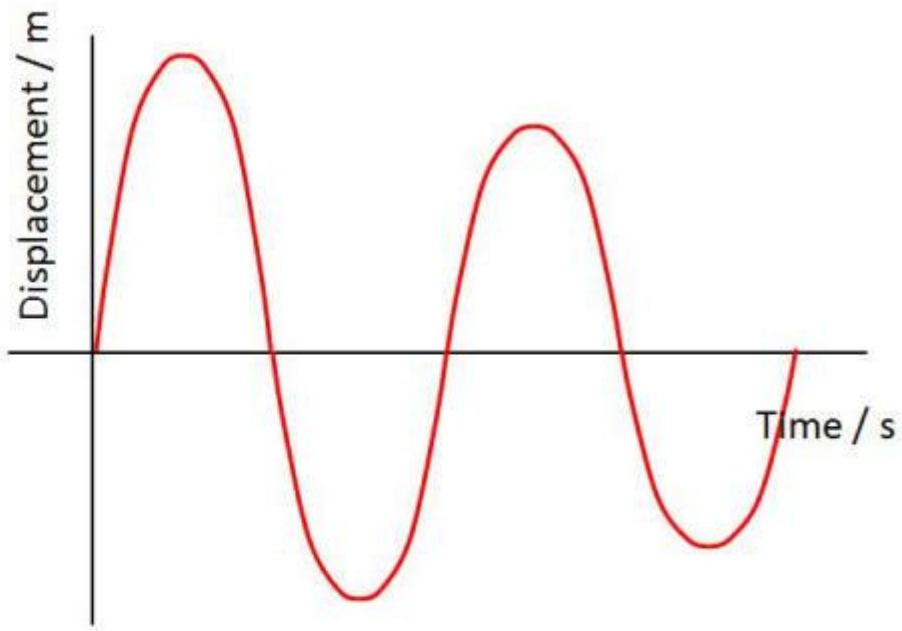

Figure 27 A bungee rope would result in the climber bouncing up and down.

There would be every possibility of the climber being injured as he (or she) bounces about.

The idea of a climbing rope is that it should be strong enough to hold the climber's weight during climbing, but when the climber has a fall, the rope plastically deforms to allow the climber to slow down to zero speed without suffering injury (*Figure 28*).

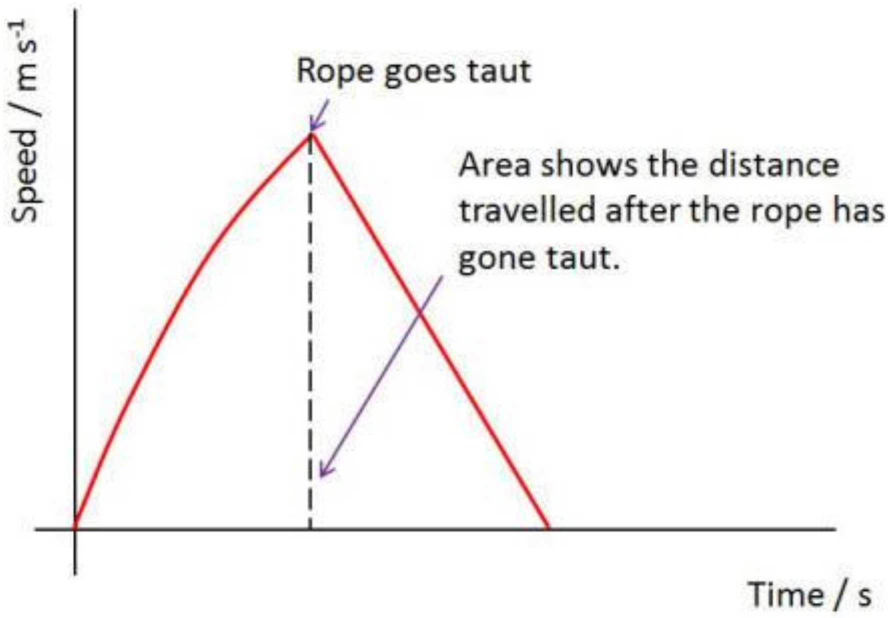

Figure 28 Rate of slowing down is much lower in a proper climbing rope

The energy is dissipated as heat, so there are fewer oscillations of reduced amplitude. It is heavily damped (*Figure 29*).

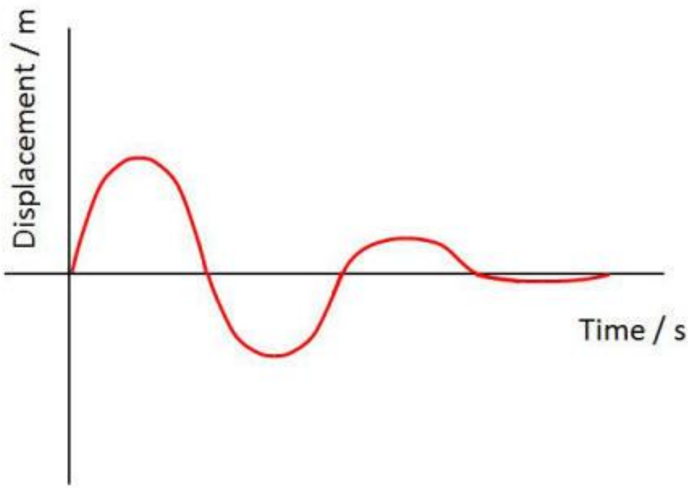

Figure 29 A climber will not bounce about so much on a climbing rope that plastically deforms

Once the rope has plastically deformed, it cannot regain its previous shape. It has to be discarded. Ideally it should be taken home and put in the bin. It should not be left on the cliff-face as it would constitute an environmental hazard as well as being unsightly litter. It should certainly NOT be used again for climbing.

Tutorial 8.03 Questions

8.03.1

A rope hanging from a tree swings with a period of 5 s. What is its frequency?

8.03.2

Look at *Figure 15*.

What is the phase relationship between these two waves? Which one is lagging?

8.03.3

What is the difference between natural and forced oscillations?

8.03.4

At a certain engine speed, a car's wing mirror starts to vibrate strongly. Why does this happen?

8.03.5

Explain why worn shock absorbers can make a car fail its annual MOT (an annual check in which about 30 safety items are checked. It is illegal to drive a car without a current MOT)

8.03.6

Sketch graphs to show heavy damping and critical damping.

8.03.7

Explain why a car shock absorber needs to be a critically damped system rather than an over-damped system.

8.03.8

Two students are told that when damping occurs, the amplitude decays as a constant fraction of the amplitude of the previous oscillation. Describe how they could investigate this.

Tutorial 8.04 Simple Harmonic Motion

All Syllabi

Contents

There are four different kinds of **motion** that we can encounter in Physics:

- Linear (in a straight line)
- Circular (going round in a circle)
- Rotational (spinning on an axis)
- Oscillations (going backwards and forwards in a to-and-fro movement.)

Anything that swings or bounces or vibrates in a regular to-and-fro motion is said to **oscillate**. Examples include a swinging pendulum or a spring bouncing up and down. It is said that the regularity of a swinging object was first described by a teenage Galileo who watched a chandelier swinging during a church service in Pisa.

8.041 Simple Harmonic Motion

Simple Harmonic Motion (SHM) describes the way that **oscillating objects** move. Consider a spring with a mass going from side to side. A mass is mounted on a small railway truck, which is free to move from side to side, and there is negligible friction in the truck. The system is perfectly horizontal so that we don't have to worry about gravity (*Figure 30*).

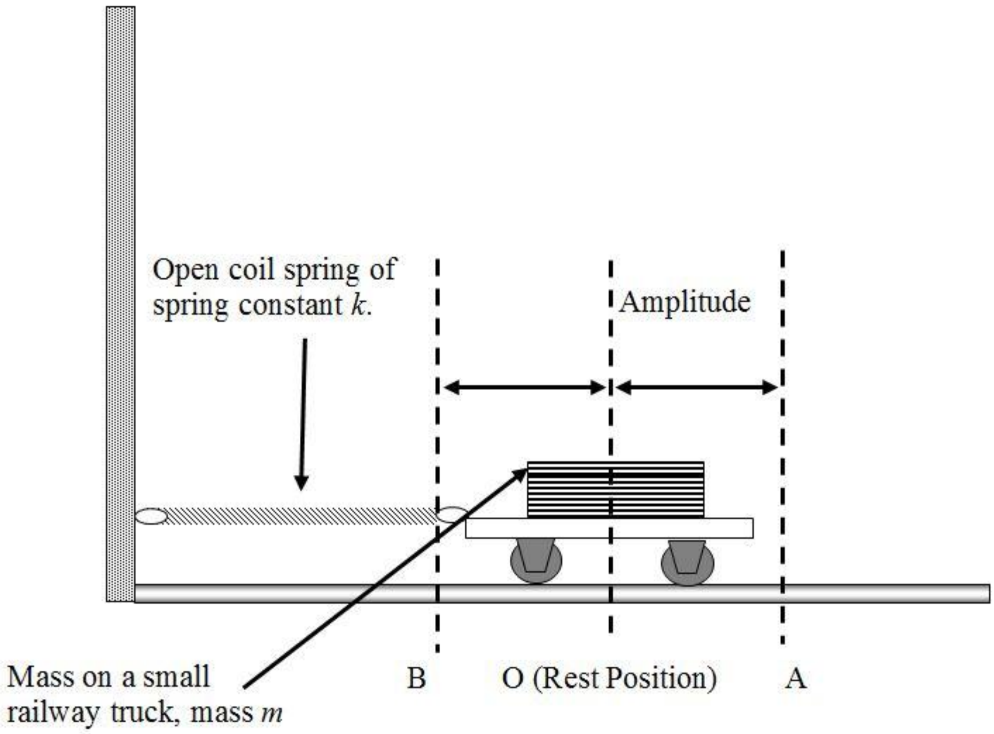

Figure 30 Explaining SHM

The rest or **equilibrium position** at **O** is where the spring would hold the mass when it is not bouncing. **A** is the position where the spring is stretched the most, and **B** is where the spring is squashed most.

- At **A** there is a large **restoring force** because that is where the spring is stretched most.
- As a result of this the mass is accelerated. It accelerates towards the rest position.

When the truck is released:

- Its velocity to the left increases.
- The acceleration decreases as the mass approached the rest position.
- Because of inertia, the mass overshoots the rest position.
- Then the spring is being compressed, and there is a restoring force to the right.
- At **B** the acceleration is at a maximum, but this time to the right.

At both **A** and **B**, the **potential energy** is at a **maximum**; the **kinetic energy** is **zero**.

As the mass passes the equilibrium position, there is zero potential energy, but maximum kinetic energy because this is the point at which the object has its greatest velocity (upwards or downwards).

Therefore, there is an **interchange** between **potential** and **kinetic** energy. The process is never 100 % efficient; some energy is lost as heat, and the process is not indefinite.

We can write down a relationship between the acceleration, a, and the displacement, x.

$$F = ma$$ .... *Equation 34*

and

$$F = kx$$ .... *Equation 35*

Therefore

$$a = \frac{F}{m} = \frac{kx}{m}$$ .... *Equation 36*

So, we are saying that the acceleration is proportional to the displacement from the equilibrium position. However, that is not the whole story. Acceleration is a vector, so we must be careful of the direction. The acceleration is **towards** the equilibrium position.

For all cases:

If the acceleration of a body is directly proportional to its distance from a fixed point and is always directed towards that point, the motion is simple harmonic.

In code we can write:

$$a \propto -x$$ .... *Equation 37*

Therefore:

$$a = -kx$$ .... *Equation 38*

where k is a constant.

The **minus sign** is important as it tells us that the acceleration is towards the equilibrium position.

Since force, F, is directly proportional to acceleration by Newton II, we can also write:

$$F = -kx$$ .... *Equation 39*

This is **Hooke's Law**. With a bouncing spring, this is obvious. However, it is true for any system that oscillates with simple harmonic motion.

8.042 Some useful relationships for SHM

These relationships are derived by linking SHM to circular motion. Review Circular Motion if you need to.

Generally, we measure the **period**, which is the time taken to make a complete oscillation or cycle. The frequency is the reciprocal of the period

$$f = \frac{1}{T}$$ .... *Equation 40*

Acceleration can be linked to displacement by:

$$a = -(2\pi f)^2 x$$ .... *Equation 41*

This satisfies the condition for SHM that $a = -kx$; in this case $k = (2\pi f)^2$. A useful little dodge here is that π^2 » 10.

Angular velocity is a quantity that is borrowed from circular motion. It is sometimes called **angular frequency**. It is the angle turned per second. In SHM terms, we can consider it as the fraction of a cycle per second. It can be, of course, greater than 1:

$$\omega = 2\pi f$$ .... *Equation 42*

In some texts you may see the equation for acceleration in SHM written as:

$$a = -\omega^2 x$$ .... *Equation 43*

The **speed** at any point in the oscillation given by:

$$v^2 = (2\pi f)^2(A^2 - x^2)$$ .... *Equation 44*

$$\Rightarrow v^2 = 4\pi^2 f^2(A^2 - x^2)$$ .... *Equation 45*

$$\Rightarrow v = 2\pi f\sqrt{(A^2 - x^2)}$$ .... *Equation 46*

In this relationship, A is the **amplitude,** and x is the **displacement** from the equilibrium position. If $x = 0$, v has a maximum value; if $x = A$, $v = 0$. The velocity is 0 at each extreme of the oscillation. So, we can rewrite the equation as:

$$v_{max} = 2\pi f A$$ .... *Equation 47*

Note that the relationship only gives the **speed** (the magnitude of the velocity). This is because the displacement is squared, so the minus sign disappears. The relationship that gives **velocity** is:

$$v = -A\omega\, sin(\omega t)$$ .... *Equation 48*

The **displacement**, x, is given by:

$$x = \pm A\, cos(\omega t)$$ .... *Equation 49*

Remember that $\omega = 2\pi f$, *Equation 42*.

8.043 Graphical Treatment of SHM

The displacement can be shown graphically (*Figure 31*).

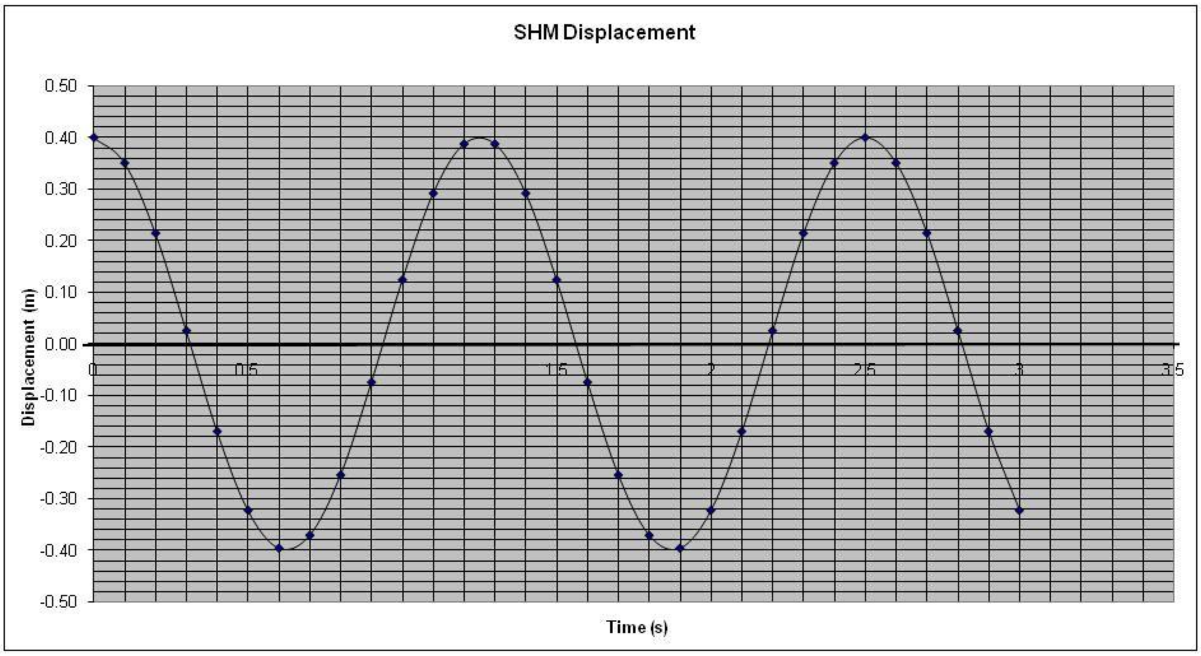

Figure 31 Displacement time graph for a simple harmonic oscillator

Note that the displacement follows the **cosine** function. That is because we have to start the oscillator by displacing it. It won't do it from the rest position.

The plus and minus sign here tells us that the motion is forwards and backwards. Which sign we give for direction is up to the individual. Generally left to right is forwards. All these equations are true for any simple harmonic motion. We can show the relationships **graphically** by showing displacement, velocity, and acceleration against time (*Figure 32*):

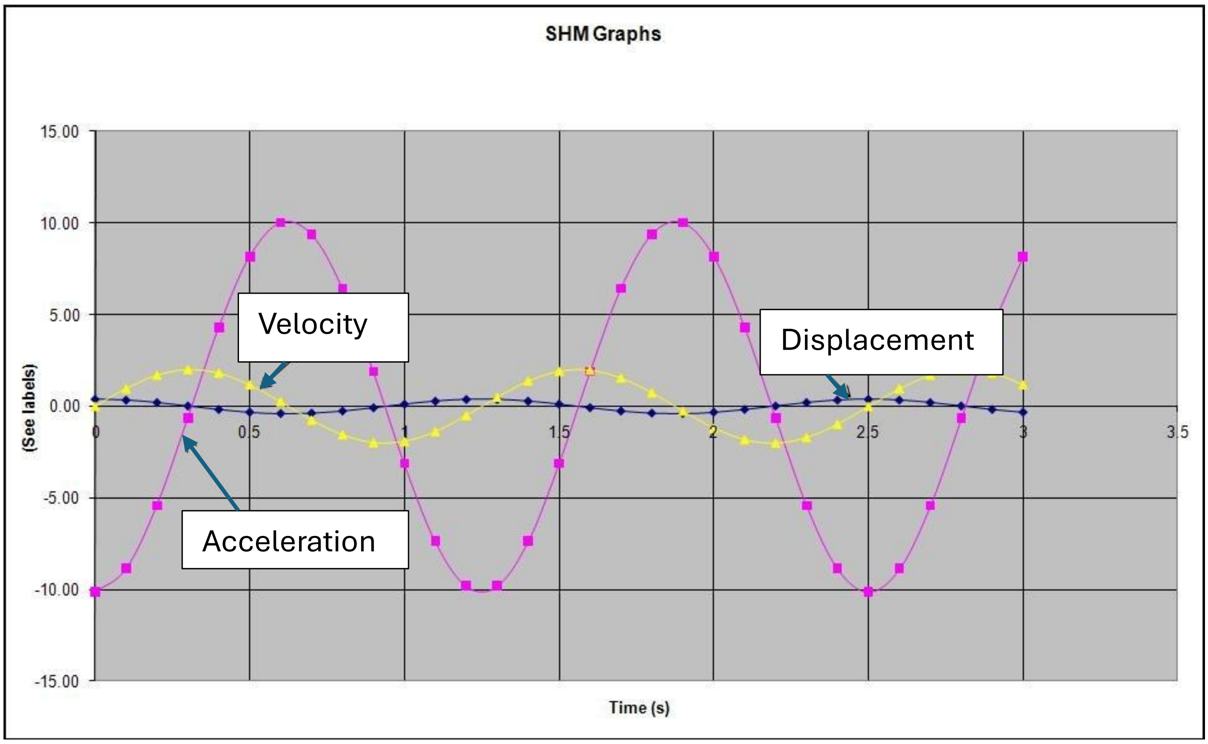

Figure 32 graphs of acceleration, velocity and displacement against time for a simple harmonic oscillator

These graphs are **sinusoidal**. The displacement is $\pi/2$ radians (90 ° or ¼ cycle) behind the velocity. The displacement and acceleration are p radians out of phase.

Worked Example A particle moving with simple harmonic motion has velocities 4 cm s^{-1} and 3 cm s^{-1} at distances of 3 cm and 4 cm respectively from the equilibrium position. What is the amplitude of the oscillation? What is the velocity of the particle as it passes the equilibrium position?
Answer We know that $v^2 = 4\pi^2 f^2(A^2 - x^2) \Rightarrow v = 2\pi f\sqrt{(A^2 - x^2)}$ [A - amplitude, x – displacement] When x = +3 cm, v = 4 cm s^{-1}; when x = + 4 cm, v = 3 cm s^{-1}. We don't know what f is. We can substitute the numbers into the equations: $4^2 = 4\pi^2 f^2(A^2 - 3^2)$ ...... [1] $3^2 = 4\pi^2 f^2(A^2 - 4^2)$ ...... [2]
To get rid of the $4\pi^2 f^2$ we need to divide [1] by [2]: $\frac{16}{9} = \frac{A^2 - 9}{A^2 - 16}$
Rearranging: $\Rightarrow 16(A^2 - 16) = 9(A^2 - 9)$ $\Rightarrow 16A^2 - 9A^2 = 256 - 81$ $\Rightarrow 7A^2 = 175$ $\Rightarrow A^2 = 175 \div 7 = 25$ $\Rightarrow A =$ **<u>5 cm</u>**
Now we can find the period by finding ω. Since $\omega = 2\pi f$, we can rewrite the equation $v^2 = 4\pi^2 f^2(A^2 - x^2)$ as $v^2 = 4\omega^2(A^2 - x^2)$: $16 = \omega^2 (25 - 9) = 1$ rad s^{-1} $\Rightarrow T = 2\pi \div 1 =$ **<u>6.28 s</u>**
Now we can work out the velocity at the equilibrium point (x = 0). $v^2 = 1(25 - 0) = 25$ $\Rightarrow v =$ **<u>5 cm s^{-1}</u>**

You must make sure your calculator is set to **radians**.

8.044 Calculus Treatment of SHM (Extension)

Consider the system at the start of the tutorial, the small railway truck on a perfectly horizontal track. The total mass of the truck is m kg, the maximum displacement is x m, and the spring constant of the spring is k N m^{-1}. There is negligible friction (*Figure 33*).

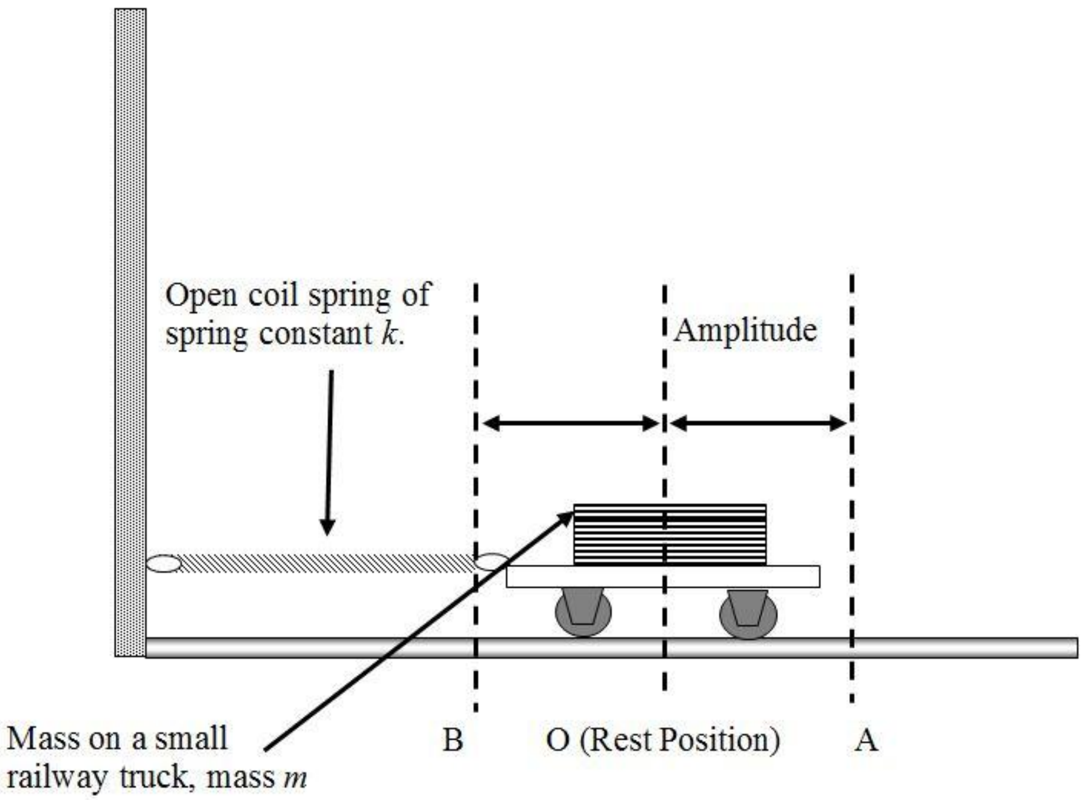

Figure 33 Small truck on horizontal track

From Hooke's Law we know that to displace the truck by x m to the right, we have to apply a force of F N. We can say that:

$$F = kx \quad \text{............ Equation 50}$$

Since the force from the spring is towards the rest position (i.e. from right to left), we need to take account of the direction by adding a minus sign:

$$F = -kx$$ ………… *Equation 51*

We also know from Newton II that:

$$F = ma$$ ………… *Equation 52*

From linear motion we know that:

$$a = \frac{dv}{dt} = \frac{d_2x}{dt^2}$$

………… *Equation 53*

We are using x as code for displacement rather than s.

We combine this with Newton II to give:

$$F = ma = m\frac{d_2x}{dt^2}$$

………….. *Equation 54*

We also combine this with Hooke's Law equation to write:

$$m\frac{d_2x}{dt^2} = -kx$$

……….. *Equation 55*

And this rearranges to:

$$\frac{d_2x}{dt^2} = \frac{-kx}{m}$$

………… *Equation 56*

The solution to this requires the mathematical concept of complex numbers ($i^2 = -1$) and de Moivre's Theorem (No, I have not heard of it before and yes, I looked it up) and is beyond what we need to consider at this level.

Therefore, we write the solution to the second order differential equation as:

$$x = A\cos(\omega t)$$

.... *Equation 57*

Strictly speaking we should add a constant, k:

$$x = A\cos(\omega t + k)$$ .... *Equation 58*

This equation shows that the displacement is positive. By convention, we start the oscillation with a positive displacement. Work has to be done to make the initial disturbance.

Note that other sinusoidal waveforms are often described by the equation:

$$x = A\sin(\omega t)$$ .... *Equation 59*

By convention these waveforms start at zero displacement. Watch out for the context of the sinusoidal waveform.

Maths Note

The differential of $\cos Kx$ is given by the general equation:

$$\frac{d}{dx}\cos(Kx) = -K\sin Kx$$

where K is a constant.

Don't forget that the expression has to be multiplied by K, the constant.

Similarly, the derivative of $-K \sin x$ is this:

$$\frac{d}{dx}(-K \sin Kx) = -K^2 \cos Kx$$

The constant represented by K in the maths note is ω, the angular velocity. The velocity can be worked out:

$$v = \frac{dx}{dt} = -A\omega \sin(\omega t)$$

.... *Equation 60*

And the acceleration:

$$a = \frac{dv}{dt} = \frac{d_2x}{dt^2} = -A\omega^2 \cos(\omega t)$$

.... *Equation 61*

You must make sure your calculator is set to **radians**.

Tutorial 8.04 Questions

8.04.1

Look at *Figure 30*. Which way is the restoring force? Why is there acceleration? In which direction is the acceleration?

8.04.2

Write down the formulae that describe kinetic energy and the elastic potential energy in a spring.

(The latter formula is NOT $E_p = mg\Delta h$).

8.04.3

How is the statement on energy interchange on Page 39 consistent with the Law of Conservation of Energy?

8.04.4

What do you think would happen if the direction of the acceleration on Page 39 were away from the rest position?

8.04.5

A pendulum has a period of 3.0 s and an amplitude of 0.10 m. What is its frequency and what is its maximum acceleration?

8.04.6

A punch-bag of mass 0.60 kg is struck so that it oscillates with SHM. The oscillation has a frequency of 2.6 Hz and an amplitude of 0.45 m. What is:

(a) the maximum velocity of the bag?

(b) the maximum kinetic energy of the bag?

(c) What happens to the energy as the oscillations die away?

8.04.7

A simple harmonic oscillator has a frequency of 3.0 Hz, and an amplitude of 0.080 m.

(a) Calculate the angular velocity and give the correct units.

For a time of 2.3 s, work out:

(b) the displacement.

(c) the velocity.

(d) the acceleration.

Give your answer to an appropriate number of significant figures.

Tutorial 8.05 Simple Harmonic Systems	
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8.051 Mass on a spring	8.052 The Simple Pendulum
8.053 Energy in Simple Harmonic Motion	

8.051 Mass on a spring

The extension of a spring is directly proportional to the force (Hooke's Law). Consider a mass, m, put onto a spring of spring constant k so that so that it stretches by an extension l (*Figure 34*).

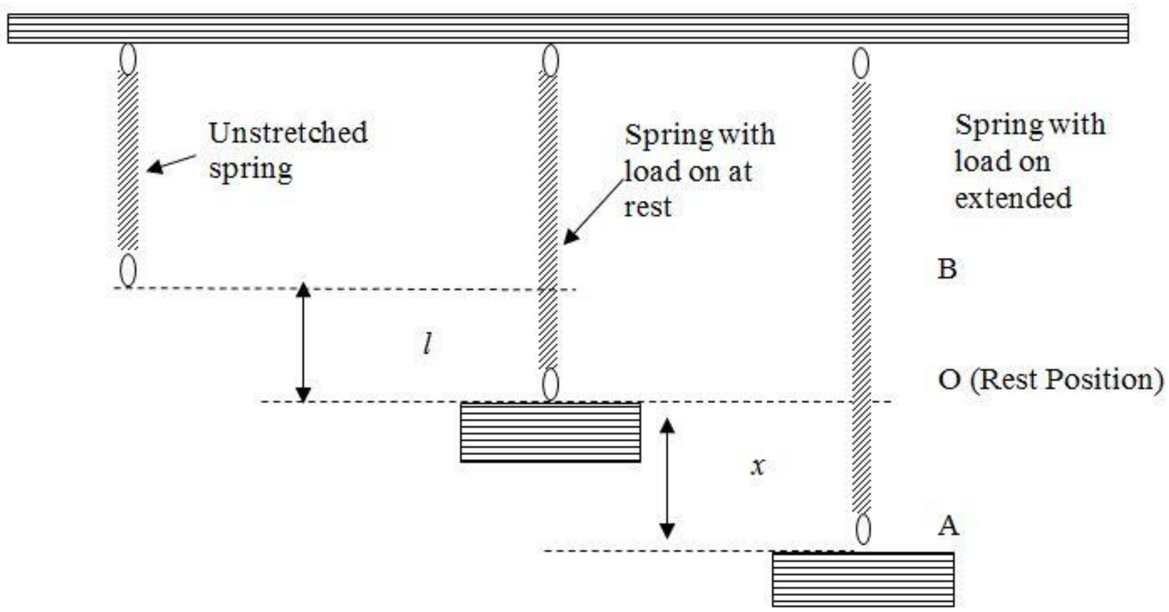


Figure 34 Mass on a spring oscillator

The force on the spring = mg, and the stretching tension = kl.

$$\Rightarrow mg = kl$$ .... *Equation 62*

Suppose the spring is pulled down by a distance x below the rest position. Now the stretching force become $k(l + x)$. This is also the tension in the spring acting **upwards**. So, the restoring force, $F_{up} = k(l + x) - mg$. This is because mg is the weight, which always acts **downwards**.

Since $kl = mg$, we can write:

$$F_{up} = kl + kx - kl = kx$$ ………… *Equation 63*

We can now apply Newton II to write:

$$-kx = ma$$ .... *Equation 64*

(The negative sign tells us that the force is upwards)

We know from SHM that $a = -(2\pi f)^2 x$, so we can write:

$$a = -\frac{kx}{m} = -(2\pi f)^2 x$$

.... *Equation 65*

So, we can tidy *Equation 65* up to say that:

$$(2\pi f)^2 = \frac{k}{m}$$

.... *Equation 66*

Since $a = -(2\pi f)^2 x$, we can say that the condition for SHM is satisfied in this system, as long as Hooke's Law is obeyed. We can now rearrange *Equation 66* above to write:

$$f^2 = \frac{1}{4\pi^2}\left(\frac{k}{m}\right)$$

.... *Equation 67*

This then becomes:

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{k}{m}\right)}$$

.... *Equation 68*

Since:

$$T = \frac{1}{f}$$

.... *Equation 69*

we can now write down an expression to relate the period with the mass and the spring constant:

$$T = 2\pi\sqrt{\left(\frac{m}{k}\right)}$$

.... *Equation 70*

This tells us that if we want to double the period, the mass has to be increased by four times.

If we plot a graph of T^2 against m we will get a straight line, since $T^2 = 4\pi^2 (m/k)$:

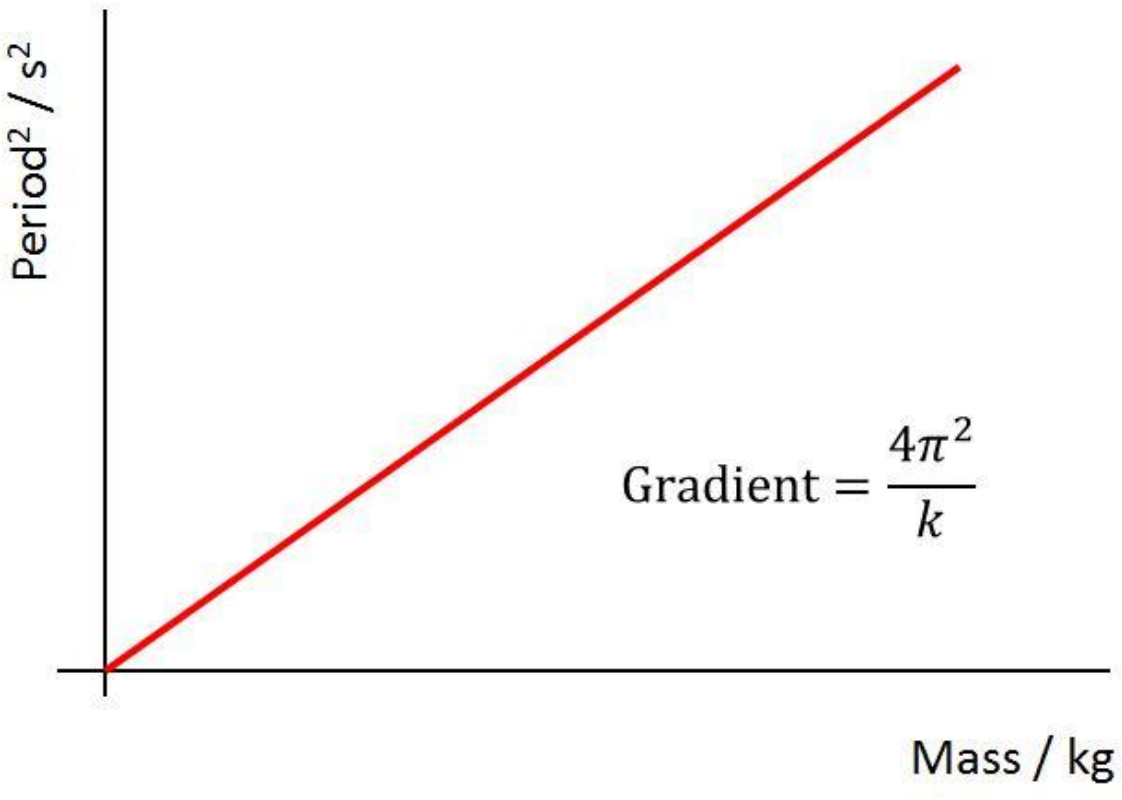


Figure 35 Graph of Period^2 against mass

The gradient will be $4\pi^2/k$ which we can approximate to $40/k$, since $\pi^2 \approx 10$.

This is a part of a **required practical**.

The relationship of the graph suggests that the line should cut through the origin. However, we may find that it does not. This is due to the mass of the spring itself; the effective mass of the spring is about 1/3 the actual mass of the spring itself. However, if the mass on the spring is very much bigger than the mass of the spring, this effect is negligible.

Worked Example

A light spiral spring is loaded with a mass of 50 g and extends by 10 cm. What is the period of small vertical oscillations if the acceleration due to gravity is 9.8 m s^{-2}?

Answer

We need to work out the spring constant using Hooke's Law $F = ke$:

$$k = F/e = \frac{0.05 \text{ kg} \times 9.8 \text{ m s}^{-2}}{0.1 \text{ m}} = 4.9 \text{ N m}^{-1}$$

Now we can use $T = 2\pi \sqrt{(m/k)}$ to work out the period:

$$T = 2\pi\sqrt{\left(\frac{0.05 \text{ kg}}{4.9 \text{ N m}^{-1}}\right)} = 2\pi\sqrt{(0.0102 \text{ s}^2)} = 2 \times p \times 0.1010 \text{ s} = 0.6346 \text{ s} = \mathbf{0.63 \text{ s}} \text{ (2 s.f.)}$$

A common bear trap is to forget to take the square root.

8.052 The Simple Pendulum

Consider a small bob of mass m hanging from a very light string, length l, which in turn hangs from a fixed point (*Figure 36*). If it is pulled to one side through a **small** angle θ, it will swing with a to-and-fro movement in the arc of a circle. We need the angle to be small, so that we can say that the arc **OA** is (nearly) the same length as the chord **OA**.

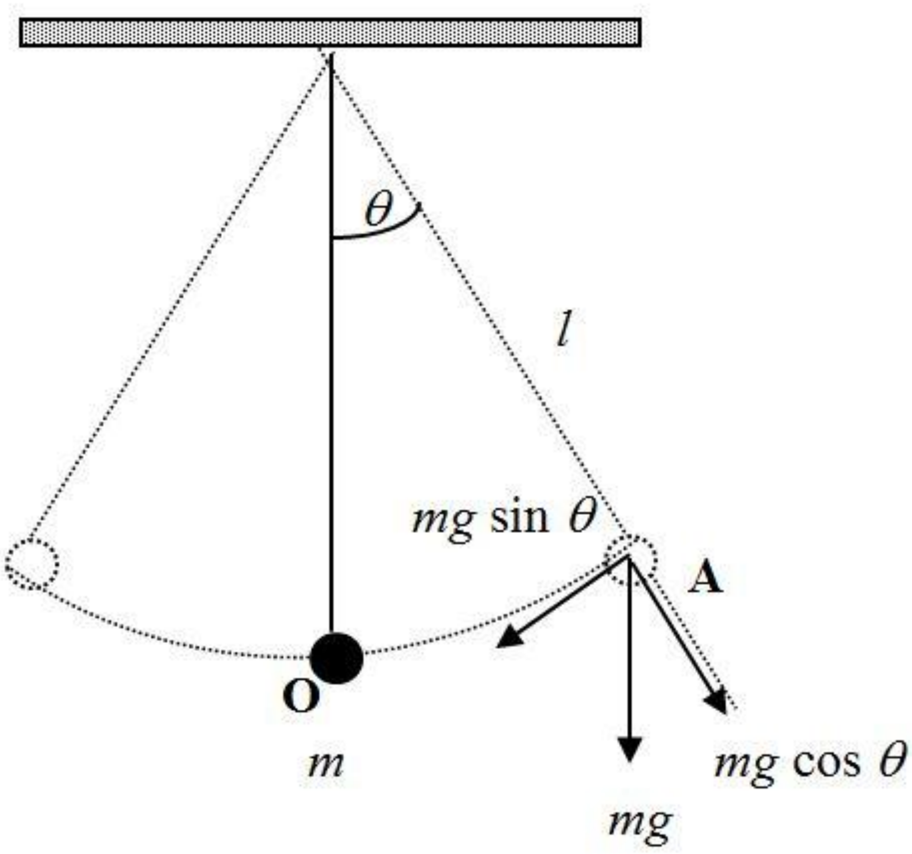


Figure 36 A simple pendulum

As the weight mg is a vector, we can break it down into its two perpendicular components, $mg \cos \theta$ and $mg \sin \theta$. Remember that the weight mg is the resultant and always points vertically downwards.

At point **A** the bob accelerates with an acceleration a due to the force $mg \sin \theta$.

We can apply Newton II to write:

$$-mg \sin \theta = ma \quad \text{.............. Equation 71}$$

[negative sign as the force is directed to equilibrium position]

If θ is small and in radians, we can say that $\sin\theta = \theta$. This does not work for degrees.

Make sure your calculator is set to radians.

We can measure the chord **OA**, which is the displacement. Remember that displacement is the straight-line distance between two points. Now we can say that the displacement:

$$s = l \sin \theta$$ .... *Equation 72*

So, we can therefore rewrite this as:

$$s = l\theta$$ .... *Equation 73*

So now we can write:

$$-mg\theta = -mg\frac{s}{l} = ma$$ .... *Equation 74*

The m terms cancel out. Therefore:

$$a = -\frac{gs}{l} = -(2\pi f)^2 s$$ .... *Equation 75*

The relationship $a = -(2\pi f)^2 s$ arises because this is a simple harmonic oscillator.

In this case the s terms cancel out to give:

$$(2\pi f)^2 = \frac{g}{l}$$

.... *Equation 76*

This rearranges to:

$$f^2 = \frac{1}{4\pi^2}\left(\frac{g}{l}\right)$$

.... *Equation 77*

This then becomes:

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{g}{l}\right)}$$

.... *Equation 78*

We know that period is given by:

$$T = \frac{1}{f}$$

.... *Equation 79*

So, we turn the equation upside down and write the formula linking period, T, with length, l, and gravity constant, g, as:

$$T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$$

.... *Equation 80*

Notice that T is **independent** of amplitude or mass of the bob. If a pendulum clock were taken to the moon, its timekeeping would be somewhat altered. Note that we have not used the angular velocity term ω. It is easier to use $2\pi f$.

If we plot a graph of T^2 against l, we get a straight line of which the gradient is $4\pi^2/g$ (the value of which would be approximately 4). To measure g we divide $4\pi^2$ by the gradient. The graph is like this (*Figure 37*).

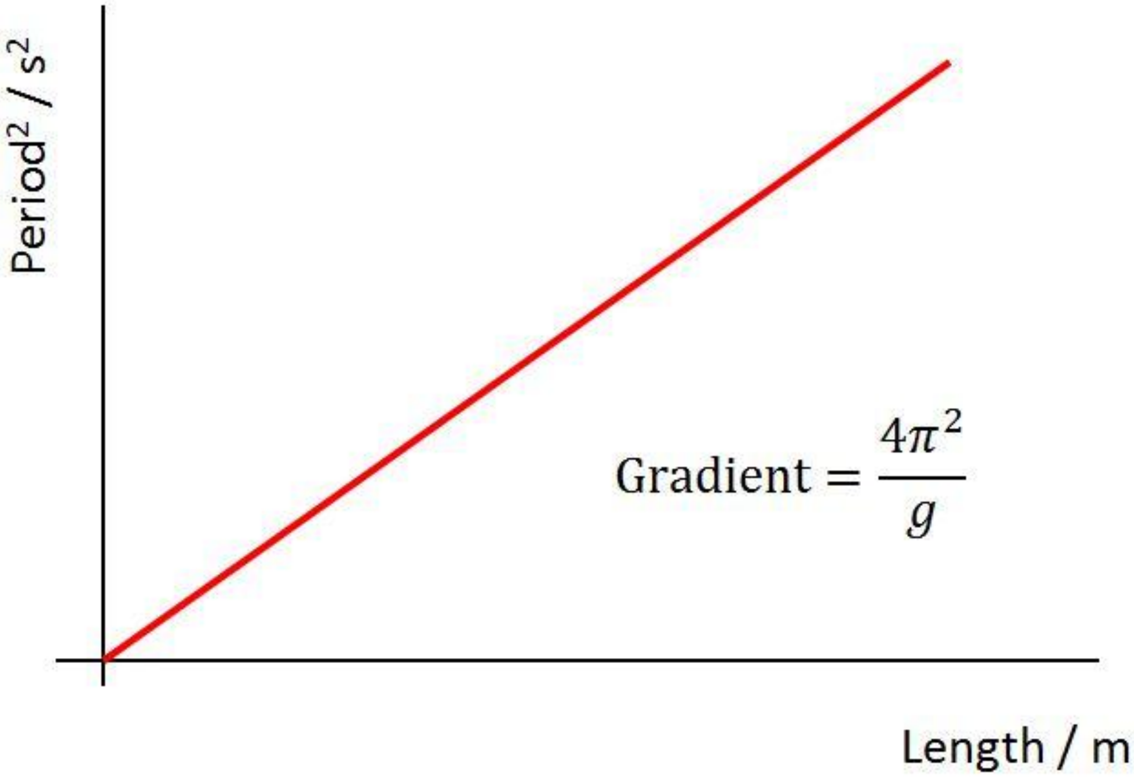


Figure 37 Graph of Period² against length is a linear progression

We can get a relatively accurate determination of this if we:

- Count at least 100 swings
- Use a swing angle of less than 10°.
- Measure *l* to the centre of the bob.
- Count the oscillations as the bob passes the centre position.

Worked Example

A simple pendulum has a period of 2.0 s and amplitude of swing of 5.0 cm. What is the maximum velocity of the bob? What is the maximum acceleration?

Answer

Velocity is at a maximum when the equilibrium position is reached.

$$v^2 = \omega^2(A^2 - x^2) \quad x = 0, A = 5.0 \text{ cm.}$$

We need to know ω.

$$\omega = 2\pi/T = 2\pi \text{ rad} \div 2.0 \text{ s} = \pi \text{ rad s}^{-1}$$

Now we can work out the velocity:

$$v^2 = \pi^2(5.0^2 - 0^2) = \pi^2 \times 25 \text{ cm} = 246.74 \text{ cm}^2\text{ s}^{-2}$$

$$\Rightarrow v = \underline{\mathbf{15.7\ cm\ s^{-1}}}$$

We can use $a = -\omega^2 s$. Acceleration is at a maximum when $s = A = 5.0$ cm

$$\Rightarrow a = -(\pi \text{ rad s}^{-1})^2 \times 5.0 \text{ cm} = \mathbf{-\ \underline{49.3\ cm\ s^{-2}}}.$$

The negative sign tells us that the direction of the acceleration is to the rest position.

8.053 Energy in Simple Harmonic Motion

There is constant **interchange** between **kinetic** and **potential** energy as the pendulum (or other oscillator) swings to-and-fro (*Figure 38*). If the system does not have to work against restrictive forces, such as friction, the total energy will remain constant.

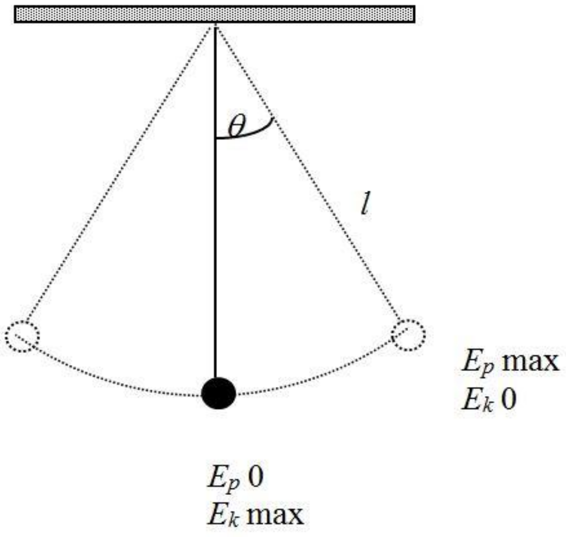


Figure 38 Interchange of energy in a pendulum.

This is the most likely level you need.

$$E_{tot} = E_p + E_k$$ ………… *Equation 81*

We can show the variation of the energy graphically (*Figure 39*).

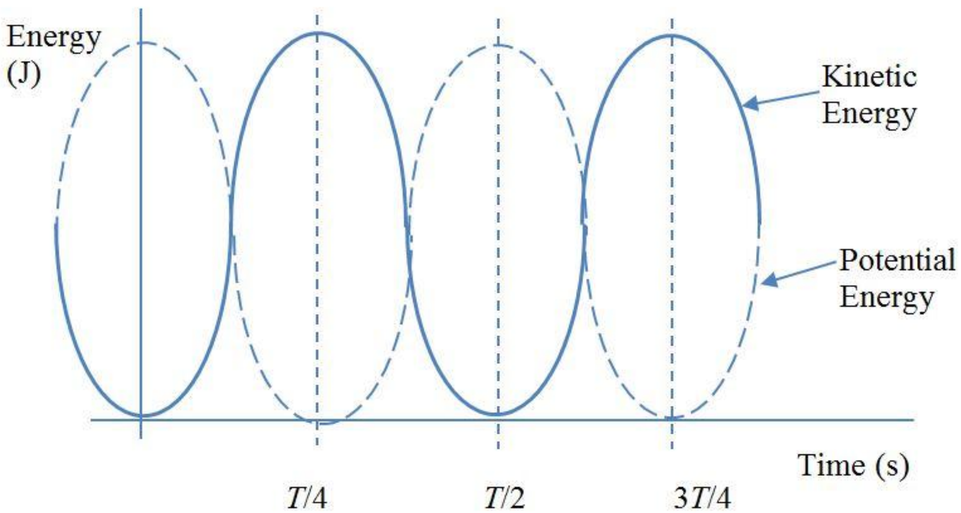


Figure 39 How potential and kinetic energy change through a cycle

Now we will look at the energy with displacement (*Figure 40*).

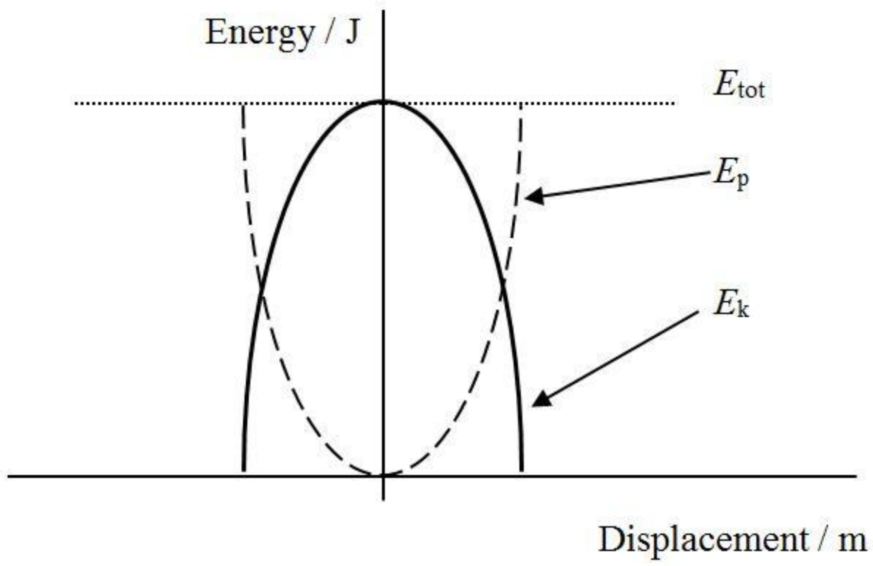


Figure 40 Energy variation with displacement

If you are not sure about this, the key points to remember are:

- Potential energy is **highest** when the oscillator is at the **maximum** amplitude.
- Kinetic energy is **highest** when the oscillator **passes the rest position**.

In the exam, you will probably be asked simply to mark on a diagram where the maximum potential and kinetic energy occur.

Tutorial 8.05 Questions

8.05.1

Why does the spring stretch when you add a mass to it?

8.05.2

A spring has a spring constant of 80 N m^{-1}. A mass of 0.5 kg is placed on the spring, and the spring is allowed to oscillate. What is the frequency of the oscillation?

8.05.3

How much would the time keeping of a pendulum clock be affected by taking it to the moon? Gravity on the moon is 1.6 N kg^{-1}, compared with 9.8 N kg^{-1} on earth.

8.05.4

Would a bouncing spring oscillator be affected by the weakened gravity field on the Moon?

8.05.5

What is the time period for a pendulum of length 4.6 m. Take g = 9.8 m s^{-2}

8.05.6

The amplitude of the swing of the pendulum in question 5 is 0.50 m. What is the maximum acceleration? What is the maximum velocity?

Tutorial 8.06 Energy in SHM
All Syllabi
Contents

8.061 Energy in SHM

We have seen how we can work out the velocity of a simple harmonic oscillator at any given point by:

$$v^2 = (2\pi f)^2(A^2 - x^2)$$ ………… *Equation 82*

We also know that **kinetic energy** is given by:

$$E_k = \frac{1}{2}mv^2$$ .... *Equation 83*

so, it doesn't take a genius to see that *Equations 82* and *83* can be combined to give:

$$E_k = ½\ m(2\pi f)^2(A^2 - x^2)$$ …………… *Equation 84*

We can see that the maximum **potential energy** is found at each end of the swing of a pendulum. As the pendulum swings away from the rest position, work is done against the restoring force. If $x = 0$, the equilibrium position, the restoring force is 0 as well. At any displacement *x* we can find the restoring force F. We can do this by using Newton II, $F = ma$ and $a = (2\pi f)^2x$.

The force at any displacement x is given by:

$$F = m(2\pi f)^2x$$ .... *Equation 85*

We know that work done = force × distance moved in the direction of the force. However, the force is not constant, and we need to consider the **average force,** which is half the maximum force.

Work done = average force × displacement

$$F_{av} = ½\, m(2\pi f)^2 x \times x = ½\, m(2\pi f)^2 x^2$$ ………… *Equation 86*

We can work out the potential energy at any point using:

$$E_p = ½\, m(2\pi f)^2 x^2$$ ………. *Equation 87*

The **total energy** at any point is simply the sum of the potential and kinetic energy:

$$E_{tot} = E_p + E_k$$ ……….... *Equation 88*

$$E_{tot} = ½\, m(2\pi f)^2(A^2 - x^2) + ½\, m(2\pi f)^2 x^2$$ ………. *Equation 89*

$$= ½\, m(2\pi f)^2 A^2$$ ……...... *Equation 90*

We can also write this as:

$$E_{k\max} = \frac{1}{2} m\omega^2 A^2$$

.... *Equation 91*

Tutorial 8.06 Questions

8.06.1

A pendulum bob has a mass of 50.0 g, and the length of the very thin string is 1.55 m from its suspension point to the centre of mass of the bob. The bob is displaced by 5.0 cm to the right of the rest position and is released. The bob swings freely.

(a) Show that the period of the swing is about 2.5 s.

(b) Calculate the angular velocity of the swing.

(c) Calculate the displacement after 1.0 s, referring to the rest position.

(d) Calculate the velocity at this point, stating the direction. Give your answer to an appropriate number of significant figures.

(e) What is the kinetic energy at this point?

(f) What is the maximum kinetic energy? When would the bob have the maximum energy?

(g) Calculate the maximum force that occurs on the bob when it's at the left-hand extremity of the swing. State the direction.

Use g = 9.8 m s^{-2}.

Tutorial 8.07 SHM and Circular Motion	
Extension	
Contents	
8.071 Linking SHM and Circular Motion	8.072 Velocity of an Oscillator
8.073 Displacement of an Oscillator	

This analysis is not required for the AQA A syllabus or for other syllabi. However, it is useful to understand the relationship between the two physics principles.

8.071 Linking SHM and Circular Motion

Simple Harmonic Motion and **Circular Motion** are very closely related. Think about a turntable with a spike attached to it. Its shadow is **projected** onto a screen. As we turn the turntable, we can see the shadow moving forwards and backwards on the screen (*Figure 41*).

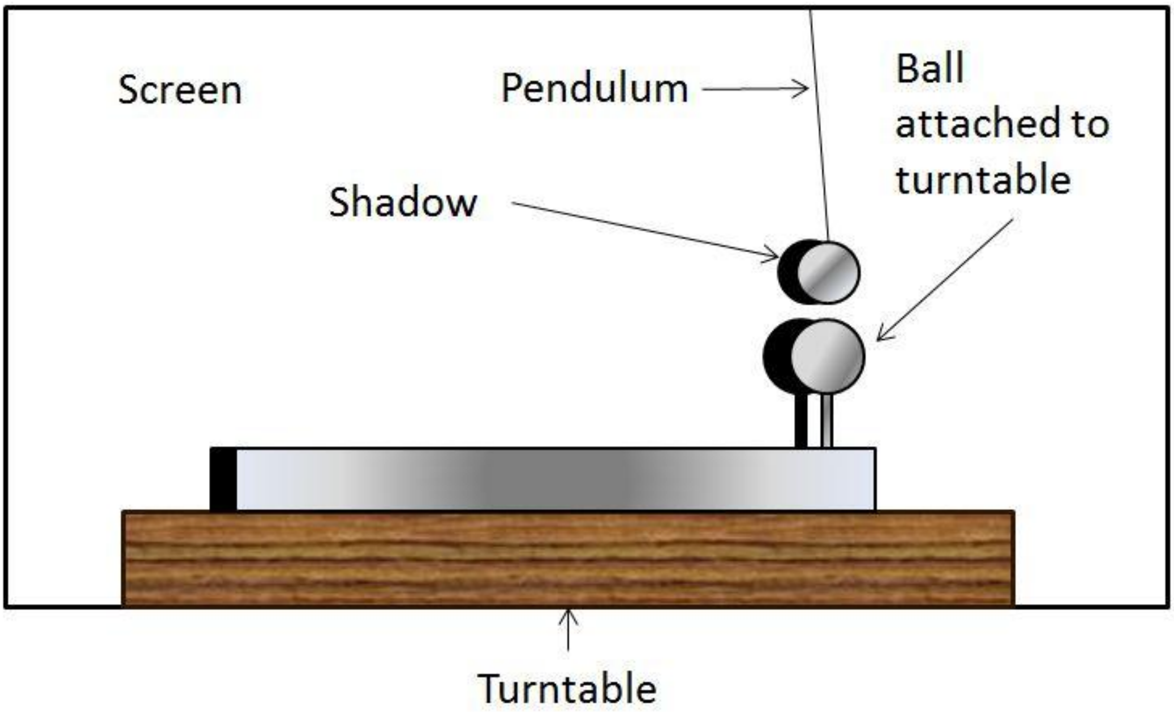


Figure 41 Comparing SHM and circular motion

We can get a pendulum to swing in phase with the ball (easier said than done).

If we look at the apparatus from above (*Figure 42*).

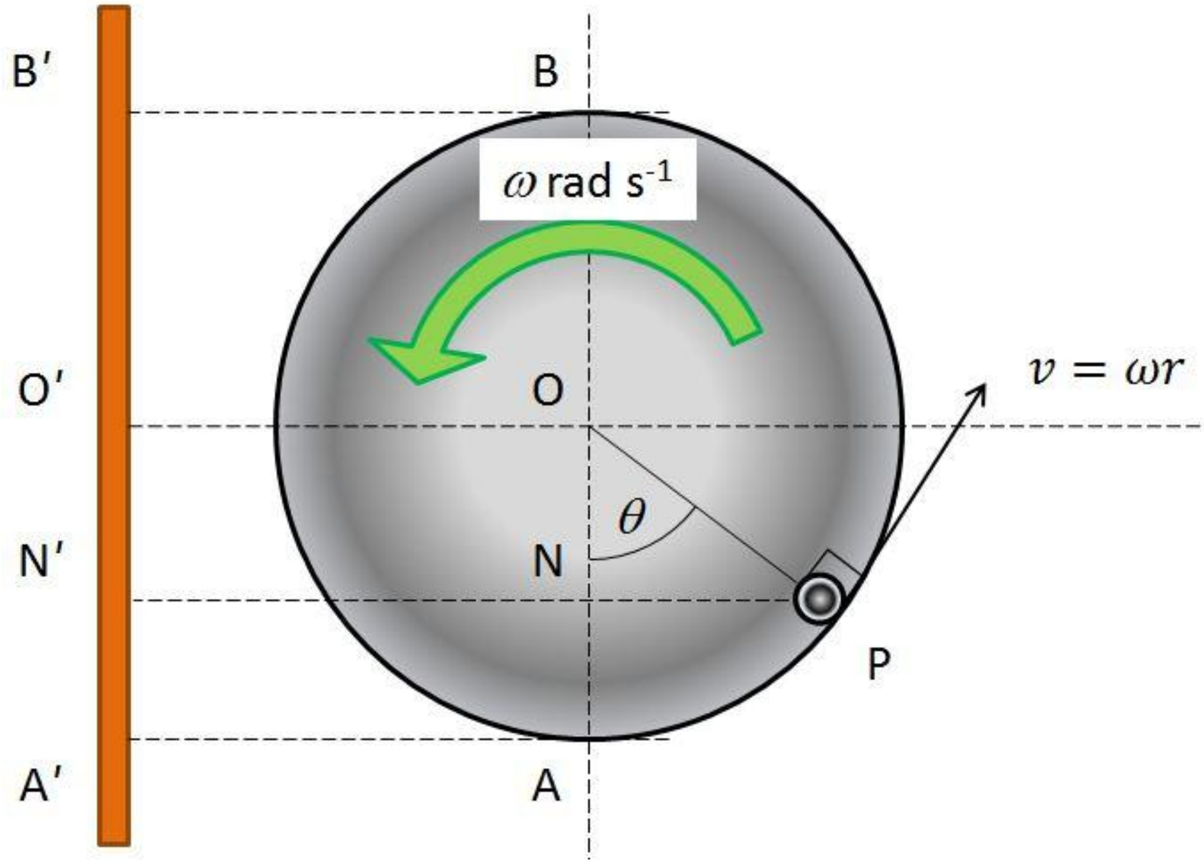


Figure 42 Showing Figure 41 from above

As **P** goes round the circumference of the revolving turntable, its projection **N** will move up the diameter of the circle **AOB**. The shadow of **P** at this instant is projected onto the screen at **N´**. The shadow will move up and down the screen along the line **A´N´O´B´**. The turntable is revolving at an angular velocity of *w* radians per second, and its linear speed is ωr m s^{-1}.

How can we show that the image of **P** describes simple harmonic motion?

Let us consider our shadow going across the screen (*Figure 43*).

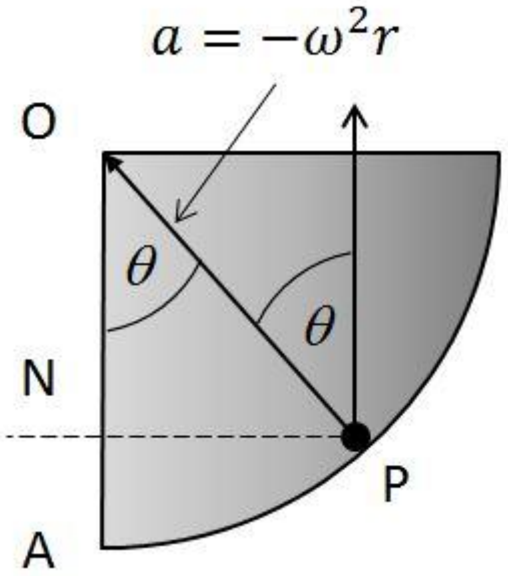


Figure 43 View from top showing the shadow describing SHM.

We know that the acceleration is towards the centre of the circle and is given by:

$$a = (2\pi f)^2 r$$ ………….. *Equation 92*

Acceleration is a vector, so has horizontal and vertical components. We can work out the acceleration parallel to **ONA** as:

$$a = (2\pi f)^2 r \cos\theta$$ ………… *Equation 93*

Since the acceleration is towards the centre, it is given a minus sign, so our formula is modified to:

$$a = -(2\pi f)^2 r \cos\theta$$ ………. *Equation 94*

If we look at the distance from the equilibrium point, we can calculate it as:

$$x = r \cos\theta$$ ………… *Equation 95*

So, we can combine *Equations 94* and *95* to give:

$$a = -(2\pi f)^2 x$$ ………… *Equation 96*

The *Equation 96* satisfies the condition for SHM that $a = -kx$; in this case $k = (2\pi f)^2$. A useful little dodge here is that $\pi^2 \approx 10$.

The time it takes for our turntable to make **one** complete revolution is called the **period**, and is given the code *T*, and is measured in seconds. It is also the time for the shadow of **P** to make **one oscillation**, or complete to-and-fro movement. We can use the simple equation

$$\text{time (s)} = \frac{\text{distance (m)}}{\text{speed (m s}^{-1})}$$

Therefore

$$\text{period (s)} = \frac{\text{circumference of the turntable (m)}}{\text{linear speed (m s}^{-1})}$$

In code:

$$T = \frac{2\pi r}{v} = \frac{2\pi r}{\omega r} = \frac{2\pi}{\omega}$$

.... *Equation 97*

Now we know that **frequency**,

$$f = \frac{1}{T}$$

...............*Equation 98*

Therefore:

$$f = \frac{\omega}{2\pi}$$

.... *Equation 99*

So that:

$$\omega = 2\pi f$$ .... *Equation 100*

In some texts you may see the equation for acceleration in SHM written as:

$$a = -\omega^2 x$$ .... *Equation 101*

T is independent of the radius of the turntable, hence the amplitude of the oscillation. If the amplitude is increased, the body travels faster, so T is not affected. This kind of oscillation is called **isochronous**, which means that it takes the same time to complete each cycle.

8.072 Velocity of an Oscillator

The direction of the **velocity** of anything moving in a circle is always at a **tangent** (*Figure 44*).

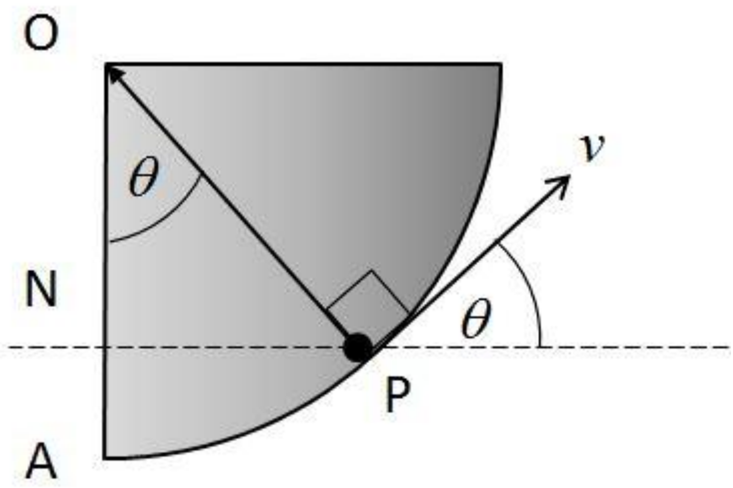


Figure 44 Velocity direction in circular motion is always tangential

So, the component of the velocity parallel to **AOB** is:

$$v_{AOB} = v \sin \theta$$ ……………. *Equation 102*

See *Figure 45*.

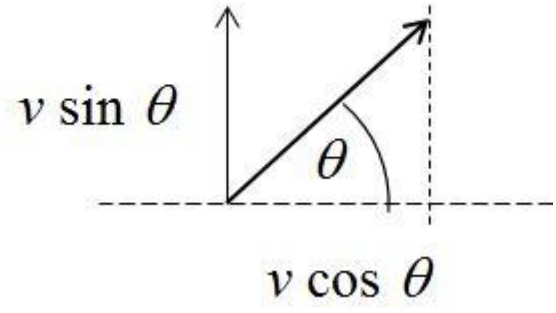


Figure 45 Components of the velocity vector

Since:

$$v = (2\pi f)r$$ .... *Equation 103*

the velocity parallel to **AOB** can be written:

$$v_{AOB} = -v \sin \theta = -(2\pi f)r \sin \theta$$ .... *Equation 104*

The negative sign tells us that the velocity is negative when the image is going upwards and positive when the image is going downwards. This ties in with the fact that the sine function has positive values for values of θ between 0 and π radians (0 – 180°) and negative values from π to 2π radians (180° – 360°).

The derivation of the equation that gives us **velocity** at any point in the oscillation is rather tedious, but the relationship is:

$$v^2 = (2\pi f)^2(A^2 - x^2)$$ ………… *Equation 105*

Therefore:

$$v = 2\pi f\sqrt{(A^2 - x^2)}$$

.... *Equation 106*

In this relationship, A is the **amplitude,** and x is the **displacement** from the equilibrium position. If $x = 0$, v has a maximum value; if $x = A$, $v = 0$. The velocity is 0 at each extreme of the oscillation.

8.073Displacement of an Oscillator

We can easily find the displacement using our circular argument. If the radius of the turntable is r, we can show quite simply that the displacement, x, is given by:

$$x = r \cos \theta$$ .... *Equation 107*

This is shown below (Figure 46).

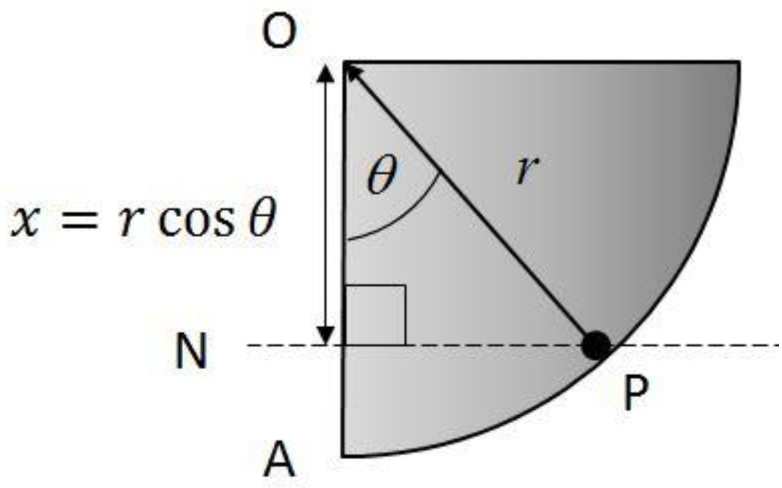


Figure 46 Displacement of the projection along a screen.

Since $\theta = \omega t$, we can rewrite *Equation 107* as:

$$x = r \cos \omega t$$ ……………...... *Equation 108*

The radius of the circle is also the amplitude, and $\omega = 2\pi f$, we can rewrite *Equation 108* as:

$$x = \pm A \cos 2\pi f t$$ ……………… *Equation 109*

The plus and minus sign here tells us that the motion is forwards and backwards. Which sign we give for direction is up to the individual. Generally left to right is forwards.

SHM and rotary (or circular) motion are also be linked in a car engine. The pistons move forward and backwards (or up and down). The connecting rod connects the piston to the crankshaft which is carrying out rotary motion. This is shown in the diagram below (*Figure 47*).

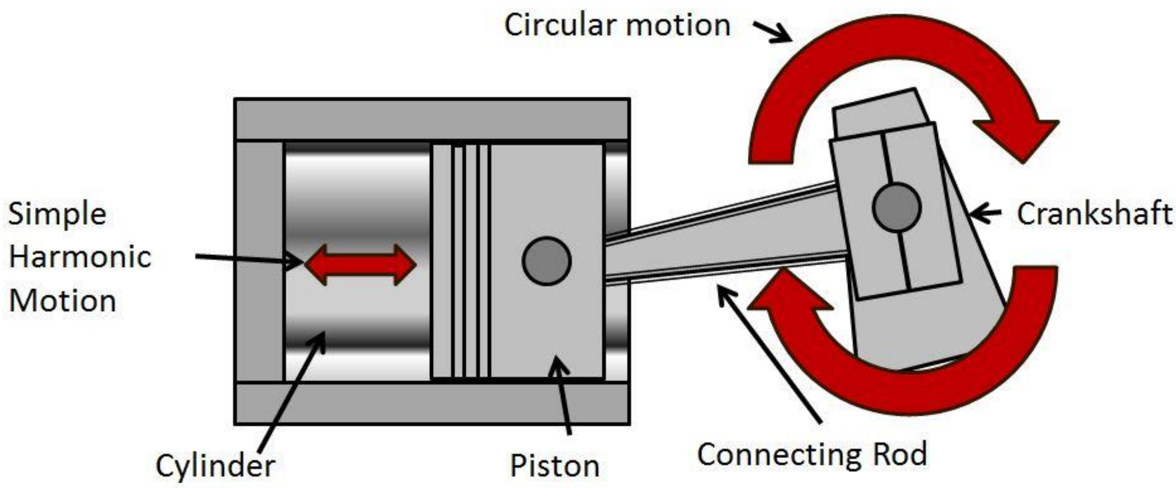


Figure 47 Linking SHM and circular motion in a car engine

Tutorial 8.07 Questions

There are no questions with this tutorial.

Answers to Questions

Tutorial 8.01

8.01.1

The object will start to accelerate vertically downwards…

although its velocity from left to right will remain the same.

The path is a parabola.

8.01.2

The velocity is always changing because…

the direction is always changing.

Acceleration = change in velocity ÷ time interval

8.01.3

Angular velocity $= v/r$ = 50 m s^{-1} ÷ 6000 m

= **0.0083 rad s^{-1}**

8.01.4

a)

1200 rpm = 20 revolutions per second

$\Rightarrow \omega = 20\ s^{-1} \times 2\pi$ = **126 rad s^{-1}**

(b)

Remember to turn the diameter to the radius.

Radius r = 35 cm ÷ 2 = 17.5 cm = 0.175 m

$\Rightarrow v = r\omega$ = 0.175 m × 126 rad s^{-1} = **22 m s^{-1}**

8.01.5

$v = \omega r$, therefore $\omega = v/r$

$a = (v^2/r^2) \times r = v^2/r$

8.016

The mass is not involved.

This is because the masses cancel out.

Tutorial 8.02

8.02.1

$$v^2 = gr \tan \theta$$

$\tan \theta$ = (18 m s^{-1})^2 ÷ (9.8 m s^{-2} × 360 m) = 0.092

θ = tan^{-1} (0.092) = 5.25°

8.02.2

(a)

$$v^2 = gr \tan \theta$$

$\Rightarrow r = v^2 \div (g \tan \theta)$ = (50 m s^{-1})^2 ÷ (9.8 m s^{-2} × tan 30) = (50 m s^{-1})^2 ÷ (9.8 × 0.577)

r = **440 m** (442 m)

(b)

$v = \omega r \Rightarrow \omega = v/r$ = 50 m s^{-1} ÷ 442 m = **0.113 rad s^{-1}**.

(c)

$F = m\omega^2 r$ = 75 kg × (0.113 rad s^{-1})^2 × 442 m = **420 N** (424 N)

8.02.3

The astronauts will end up hitting the roof.

They will stay on the roof until the negative g manoeuvre has finished.

8.02.4

$v^2 = gr$ = 9.8 m s^{-2} × 1000 m = 9800 m^2 s^{-2}

v = **99 m s^{-1}**.

8.02.5

(a)

$v = 2\pi fr$ = 2 × π × 1/6 s^{-1} × 12 m = **12.6 m s^{-1}**

(b)

a = (12.6 m s^{-1})^2 ÷ 12 m = **13.2 m s^{-2}**

(c)

R = (72 kg × [(12.6 m s^{-1})^2 ÷ 12 m]) – (72 kg × 9.8 m s^{-2}) = **247 N**

(d)

R = (72 kg × [(12.6 m s^{-1})^2 ÷ 12 m]) + (72 × 9.8) = **1658 N**

8.02.6

(a)

$v = \omega r \Rightarrow \omega$ = 7.0 m s^{-1} ÷ 0.98 m

ω = 7.14 rad s^{-1}

(b)

$a = v^2/r$ = 49 m^2 s^{-2} ÷ 0.98 m = **50 m s^{-2}**

(c)

$F = ma \Rightarrow F$ = 0.5 kg × 50 m s^{-2} = 25 N

Least tension is when acceleration = 50 m s^{-2} – 9.8 m s^{-2} = 40.2 m s^{-2}

$\Rightarrow F$ = 0.5 kg × 40.2 m s^{-2} = **20.1 N**

Tutorial 8.03

8.03.1

$f = 1/T$ = 1/5 s = **0.2 Hz**

8.03.2

Time difference t = 0.31 – 0.23 = 0.08 s.

Period T = 0.33 s

Phase = (0.08 ÷ 0.33) × 2π = **1.5 rad**

The smaller wave is lagging

8.03.3

Natural oscillations occur if you set off an oscillation…

…and let it oscillate with no further input.
A forced oscillation is one where energy is put in…
…at a frequency that is different to the natural frequency.

8.03.4

The engine is sending out vibrations
which are at the same frequency as the natural frequency of the mirror.

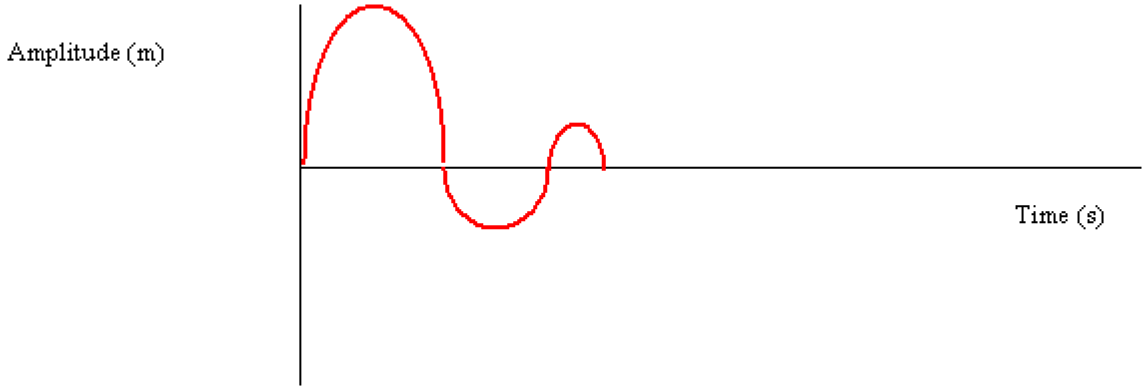
This causes the mirror to resonate.

8.03.5

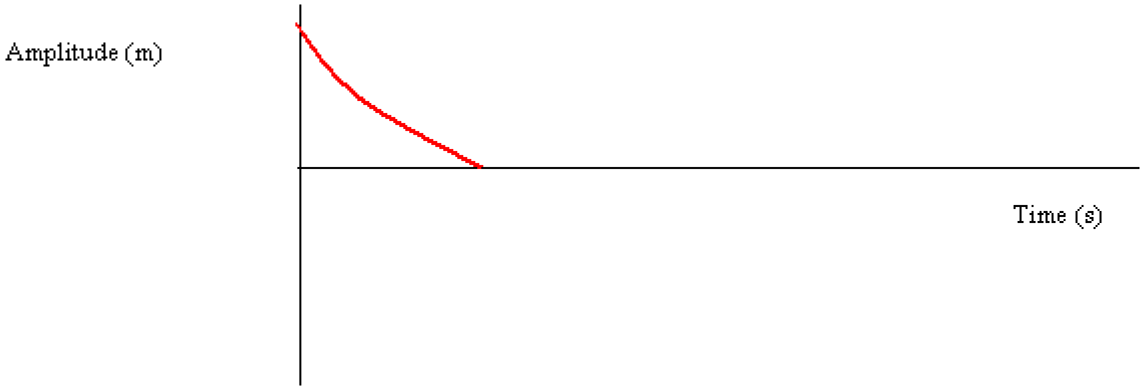
The car is dangerous to drive…
because the shock absorbers are not damping the oscillations…
caused when the car hits a bump.
If the bumps occurred at the natural frequency of the springs,
the car would bounce wildly and could go out of control.

8.03.6

Heavy damping



Critical damping



8.03.7

The suspension has to return quickly to the rest position.
If it were over-damped, the spring would still be compressed…
when the car hit the next bump, so the spring would not be able to be compressed.

8.03.8

They would need a way of measuring the amplitude, e.g. position sensor

They need to measure the amplitudes of successive oscillations.

They compare the ratio of the amplitudes.

This should remain constant.

Repeats should reduce any uncertainty

Tutorial 8.04

8.04.1

Towards the rest position
There is a force acting on the mass (Newton II)
The acceleration is towards the rest position

8.04.2

Kinetic energy:

$$E_k = ½\, mv^2$$

Elastic potential energy:

$$E_p = ½\, Fx$$

8.04.3

Energy is neither created nor destroyed; it is transferred from one form to another.

Therefore, the potential energy is turned to kinetic…

…and *vice versa*…

…and some heat.

8.04.4

The acceleration would increase directly proportionally to the displacement.
The displacement would increase as the object accelerated.
Therefore the whole system would fly apart.

8.04.5

Frequency = 1/Period = 1/3 s = 0.33 Hz

Use $a = -(2\pi f)^2 x$ = -(2 × π × 0.33 Hz)^2 × 0.10 m = (2.07 rad s^{-1})^2 × 0.10 m

a = 4.3 rad s^{-1} × 0.10 m = 0.43 m s^{-2}

8.04.6

(a)

Formula first: $v^2 = (2\pi f)^2(A^2 - x^2)$

Maximum velocity occurs when the displacement is zero

v^2 = (2 × π × 2.66 Hz)^2 × ([0.45 m]^2 – 0)

= (16.71 Hz)^2 × 0.2025 m^2

= 56.5 m^2 s^{-2}

v = **7.52 m s^{-1}**

(b)

Max $E_k = ½\, mv^2$ = ½ × 0.6 kg × (7.52 m s^{-1})^2

= 17 J

(c)

It is transferred to the surroundings as heat.

8.04.7

(a)

$\omega = 2\pi f$ = 2 × π × 3.0 Hz = 18.85 rad s^{-1}

(b)

x = 0.080 m × cos (18.85 rad s^{-1} × 2.3 s)

= -0.0648 m (i.e. below the rest position) = (-)**0.065 m**

(c)

v = -0.080 m × 18.85 rad s^{-1} × sin (18.85 rad s^{-1} × 2.3 s) = 0.885 m s^{-1} = **0.89 m s^{-1}**

(d)

a = -0.080 m × (18.85 rad s^{-1})^2 × cos (18.85 rad s^{-1} × 2.3 s) = 23.0 m s^{-2}

2 significant figures as the data are to 2 s.f.

Tutorial 8.05

8.05.1

Since the mass is being pulled by gravity there is a downward force.

8.05.2

Formula first:

$$T = 2\pi\sqrt{(m/k)}$$

T = 2 × π × (0.5 kg ÷ 80 N m^{-1})$^{0.5}$ = 2 × π × √(0.00625 s^2) = 2 × π × 0.079 s = 0.497 s

$f = 1/T$ = 1/0.497 s = 2.013 Hz = **<u>2.0 Hz</u>** (2 s.f.)

8.05.3

Let the length of the pendulum be 1.0 m

$T_e = 2\pi\sqrt{}$(l.0 m ÷ 9.8 N kg^{-1}) and $T_m = 2\pi\sqrt{}$(l.0 m ÷ 1.6 N kg^{-1})

$T_e \div T_m = 2\pi\sqrt{}$(l.0 m ÷ 9.8 N kg^{-1}) ÷ (2π √(l.0 m ÷ 1.6 N kg^{-1})) = √(9.8 N kg^{-1} ÷ 1.6 N kg^{-1})

Period on moon = √(9.8 N kg^{-1} ÷ 1.6 N kg^{-1}) that on earth = √6.125 = 2.47 times

= **<u>2.5 times</u>** (2 s.f.) i.e. it would go 2.5 times slower.

8.05.4

No

because the gravity term does not come into the equation $T = 2\pi\sqrt{(m/k)}$

8.05.5

Formula first:

$T = 2\pi\sqrt{(l/g)}$ = 2 × π × (4.6 m ÷ 9.8 m s^{-2})$^{0.5}$

<u>T = 4.3 s</u>

8.05.6

Work out f:

$f = 1/T$ = 1 ÷ 4.3 s = 0.233 Hz

$a = -(2\pi f)^2 x$ = - (2 × π × 0.233 Hz)^2 × 0.50 m = -1.072 m s^{-2}.

$v^2 = (2\pi f)^2(A^2 - x^2)$ = (2 × π × 0.233 Hz)^2 × (0.50 m^2 – 0) = 0.536 m^2 s^{-2}

v = 0.732 m s^{-1} = **<u>0.73 m s^{-1} (2 s.f.)</u>**

Tutorial 8.06

8.06.1

(a) Use:

$$T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$$

T = 2 × π × (1.55 m ÷ 9.8 m s^{-2})$^{0.5}$ = **2.499 s** = 2.50 s (QED)

(b)

ω = 2 × π × (2.50 s)$^{-1}$ = 2.513 rad s^{-1} = **2.5 rad s^{-1}** (2 s.f.)

(c) Use:

$$x = A\cos(\omega t)$$

x = 0.050 m × cos (2.513 rad s^{-1} × 1.0 s) = **-0.0405 m** (to the left of the rest position)

(d) Use:

$$v = \frac{dx}{dt} = -A\omega\sin(\omega t)$$

v = - 0.050 m × 2.513 rad s^{-1} × sin (2.513 rad s^{-1} × 1.0 s)

= -0.0739 m s^{-1} = **-0.074 m s^{-1}** (2 s.f.)

Direction is from right to left.

(You can also use $v^2 = (2\pi f)^2(A^2 - x^2)$)

v^2 = (2.513 rad s^{-1})^2 × [(0.050 m)^2 - (-0.0405 m)^2] = 5.43 × 10^{-3} m^2 s^{-2}

v = **0.074 m s^{-1}**. The answers are consistent)

(e)

Ek = 1/2 × 0.050 kg × (-0.0721 m s^{-1})^2 = **1.30 × 10^{-4} J**

(f) Use:

$$E_{k\max} = \frac{1}{2} m\omega^2 A^2$$

Ek = 1/2 × 0.050 kg × (2.513 rad s^{-1})^2 × (0.050 m)^2 = **3.95 × 10^{-4} J**

The maximum kinetic energy occurs when the bob passes through the rest position.

(g) To work out the force, we need the acceleration when the bob is at the extremities of the swing:

$a = -\omega^2 x$

a = -(2.513 rad s^{-1})^2 × -0.050 m = 0.316 m s^{-2}

Apply Newton II:

$F = ma$ = 0.050 kg × 0.316 m s^{-2} = 0.0158 N = **0.16 N** (2 s.f.) from left to right.

(Alternatively use:

$$a = \frac{dv}{dt} = \frac{d_2x}{dt^2} = -A\omega^2 \cos(\omega t)$$

a = -0.050 m × (2.513 rad s^{-1})^2 × cos (2.513 rad s^{-1} × 1.250 s) = 0.316 m s^{-2}

Apply Newton II:

$F = ma$ = 0.050 kg × 0.316 m s^{-2} = 0.0158 N = 0.16 N (2 s.f.) from left to right.)

Tutorial 8.07

8.07.1

There are no answers here, as there are no questions!